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EUCLID ±

by

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Two tales, each with a moral that may have been swamped by the spirit of the times:

Egypt's king Ptolemy Soter once asked Euclid whether there was no shorter way in geometry than that of the *Elements*. "There is no royal road to geometry", replied Euclid.

A student, after learning the very first proposition in geometry, wanted to know what he would get by learning these things. Euclid then called his slave and said, "Give him a penny since he must needs make gains by what he learns."

From Todhunter's preface to *Euclid for the Use of Schools and Colleges*

(published in 1874, more than two millennia after Euclid's death):

Numerous attempts have been made to find an appropriate substitute for the *Elements* of Euclid; but such attempts, fortunately, have hitherto been made in vain. The advantages attending a common standard of reference in such an important subject, can hardly be overestimated; and it is extremely improbable, if Euclid were once abandoned, that any agreement would exist as to the author who should replace him. It cannot be denied that defects and difficulties occur in the *Elements* of Euclid, and that these become more obvious as we examine the work more closely; but probably during such examination the conviction will grow deeper that these defects and difficulties are due in great measure to the nature of the subject itself, and to the place which it occupies in a course of education; and it may readily be believed that an equally minute criticism of any other work on geometry would reveal more and graver blemishes.

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PREFACE

Organization

The main purpose of this book is to provide prospective high school mathematics teachers with the geometrical background they need. Its core, consisting of Chapters 2 to 5, is therefore devoted to a fairly formal, that is axiomatic, development of Euclidean geometry. Chapters 6, 7, and 8 complement this with an exposition of transformation geometry. The first chapter, which introduces the teachers-to-be to non-Euclidean geometries, provides them with a perspective meant to enhance their appreciation of axiomatic systems.

The development of synthetic Euclidean geometry begins by following Euclid's *Elements* very closely. This has the advantage of convincing the students that they are learning "the real thing". It also happens to be an excellent organization of the subject matter. Witness the well-known fact that the first twenty eight propositions are all neutral. These subtleties might be lost on the typical high school student, but familiarity with Euclid's classic text must surely add to the teacher's confidence and effectiveness in the classroom. I am also in complete agreement with the sentiments Todhunter displayed in the above quoted excerpt: no other system of teaching geometry is better than Euclid's, provided, of course that his list of propositions is supplemented with a sufficient number of exercises. Occasionally, though, because some things have changed over two millennia, or else because of errors in *The Elements*, it was found advantageous to either expound both the modern and ancient versions in parallel, or else to part ways with Euclid altogether.

In order to convince the prospective teachers of the need to prove "obvious" propositions, the axiomatic development of Euclidean geometry is preceded by the informal description of both spherical and hyperbolic geometry. The trigonometric formulas of these geometries are included in order to lend numerical substance to these alternate and unfamiliar systems. The synthetic part of the course covers the standard material about triangles, parallelism, circles, ratios, and similarity; it concludes with the classic theorems that lead to projective geometry. These lead naturally to a discussion of ideal points and lines in the extended plane. Experience indicates that the non-optional portions of the first five chapters can be completed in about three quarters of a one semester course. During that time weekly homework assignments typically called for a dozen proofs.

Chapters 6 and 7 are concerned mostly with transformation geometry and symmetry. The planar rigid motions are completely and rigorously classified. This is followed by an informal discussion of frieze patterns and wallpaper designs. Inversions are developed formally and their utility for both Euclidean and hyperbolic geometry is explained.

The exposition in Chapter 8 is informal in the sense that no proofs are either offered or required. Its purpose is threefold. First there is an exposition of some interesting facts such as Euler's equation and the Platonic and Archimedian solids. This is followed by a representation of the rotational symmetries of the regular solids by means of permutations, a discussion of their symmetry groups, and a visual definition of isomorphism. Both these sections aim to develop the prospective teacher's ability to visualize three dimensional phenomena. Finally, comes the fabulous tale of Monstrous Moonshine.

Exercises

In Chapters 2 to 5 exercises are listed following every two or three propositions. This facilitates the selection of appropriate homework assignments for the professor and eliminates some of the guesswork for the students. The great majority of these exercises call for straightforward applications of the immediately preceding propositions. In the remaining and less formal chapters the exercises appear at the end of each section. Each chapter concludes with a list of review problems. Solutions and/or hints to selected exercises are provided at the end of the book.

The exercises that are interspersed with the propositions are of four types. There are relational and constructive propositions in whose answers the students should adhere to the same format that is used in the numbered propositions. The third type of exercises has to do with the alternate spherical, hyperbolic, and taxicab geometries; in these the appropriate response usually consists of one or two English sentences. The fourth, and last, type of exercises calls for the use of some computer program, and these are marked with a (C).

In the other chapters, namely 1 and 6 - 8, exercises are listed a the end of each section. The emphasis in these is on the algorithmic aspect of geometry. They mostly require the straightforward, albeit non-trivial, application of the methods expounded in the text.

Notation and conventions

Chapters 2 to 5 of this text present most of the content of Book I and selected topics from Books, II, III, IV, and VI of *The Elements*. In addition to the conventional labeling of propositions by *Chapter.Section.Number* these propositions are also identified by a parenthesized *Roman Numeral .Number* that pinpoints their appearance in Euclid's book. For example, the Theorem of Pythagoras is listed as Proposition 3.3.2(I.47).

The justifications for the various steps of the construction and proof are stated, in abbreviated form, in brackets. The abbreviations used are DFN for definition, PT for postulate, CN for common notion, and PN for proposition. The symbol $\therefore$ is used as an abbreviation for the word *therefore*. Optional sections or propositions are labeled with an asterisk. Hard exercises are so designated by a double asterisk.

Acknowledgements

The author wishes to acknowledge the many valuable suggestions and corrections provided by Mark Hunacek as well as the technical help of his colleague Pawel Szeptycki. Jack Porter taught from an early draft and provided some corrections and useful suggestions.