#1 [4 pts] Evaluate

$$\int \cos\theta\,(\sin\theta)^{4/3}\,d\theta.$$

(Hint: Use substitution.)

Substitute $u = \sin\theta$, $du = \cos\theta\,d\theta$ to obtain

$$\int u^{4/3}\,du \;=\; \frac{3}{7}u^{7/3} + C \;=\; \boxed{\frac{3}{7}(\sin\theta)^{7/3} + C.}$$

#2a [3 pts] Find the partial fraction decomposition of $\dfrac{2x-2}{x^2+2x}$**.**

The denominator $x^2 + 2x$ factors as $x(2 + x)$, so the partial fraction decomposition has the form

$$\frac{2x-2}{x^2+2x} \;=\; \frac{A}{x} + \frac{B}{x+2},$$

where A and B are constants. Cross-multiplying gives

$$\frac{2x-2}{x^2+2x} \;=\; \frac{A(x+2)+Bx}{x(x+2)} \;=\; \frac{Ax+2A+Bx}{x^2+x} \;=\; \frac{(A+B)x+2A}{x^2+x}.$$

Equating coefficients, we see that $2x - 2 = (A + B)x + 2A$, so

$$A + B = 2,$$
$$2A = -2,$$

and this system of equations has the unique solution $A = -1$, $B = 3$. So the desired partial fraction decomposition is

$$\frac{2x-2}{x^2+2x} \;=\; \boxed{\frac{-1}{x} + \frac{3}{x+2}.}$$

#2b [3 pts] Use your answer to #2a to evaluate $\displaystyle\int_2^4 \frac{2x-2}{x^2+2x}\,dx$**.**

We have by the previous problem

$$\int_2^4 \frac{2x-2}{x^2+2x}\,dx \;=\; \int_2^4 \left(\frac{-1}{x} + \frac{3}{x+2}\right)dx \;=\; \int_2^4 \frac{-dx}{x} + \int_2^4 \frac{3\,dx}{x+2}.$$

The first integral is $-\ln x]_2^4 \;=\; -\ln(4) + \ln(2) = \ln 2$. For the second integral, substitute $u = x+2$, $du = dx$ to obtain

$$\int_2^4 \frac{3\,dx}{x+2} \;=\; \int_4^6 \frac{3\,du}{u} \;=\; 3\ln u\Big]_4^6 \;=\; 3\ln 6 - 3\ln 4.$$

Putting the two pieces together, we obtain the answer

$$\boxed{3\ln 6 - 3\ln 4 + \ln 2}$$

which can be simplified to $\ln(27/16)$.

#3 [5 pts] Evaluate

$$\int_1^\infty e^{-4x}\,dx,$$

or explain why it does not exist.

In fact, the improper integral converges:

$$\begin{aligned}\int_1^\infty e^{-4x}\,dx &= \lim_{n\to\infty}\left[\int_1^n e^{-4x}\,dx\right] \\ &= \lim_{n\to\infty}\left[\left[-\frac{e^{-4x}}{4}\right]_1^n\right] \\ &= \lim_{n\to\infty}\left[-\frac{e^{-4n}}{4}+\frac{e^{-4}}{4}\right] = \boxed{\frac{e^{-4}}{4}.}\end{aligned}$$

#4 [5 pts] Use the trigonometric substitution $x = \sec\theta$ to evaluate

$$\int \frac{dx}{x^2\sqrt{x^2-1}}.$$

Substitute $x = \sec\theta$, $dx = \sec\theta\tan\theta\,d\theta$ to obtain

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2-1}} &= \int \frac{\sec\theta\tan\theta\,d\theta}{\sec^2\theta\sqrt{\sec^2\theta-1}} \\ &= \int \frac{\tan\theta\,d\theta}{\sec\theta\sqrt{\tan^2\theta}} \\ &= \int \frac{d\theta}{\sec\theta} \\ &= \int \cos\theta\,d\theta = \sin\theta + C.\end{aligned}$$

Now replacing θ with $\operatorname{arcsec} x$ yields the answer $\boxed{\sin\operatorname{arcsec} x + C}$, or equivalently $\boxed{\sin\arccos(1/x) + C.}$ This in turn can be expressed as an algebraic function:

$$\begin{aligned}\sin\arccos(1/x) + C &= \sqrt{1-\cos^2(\arccos(1/x))} + C \\ &= \sqrt{1-1/x^2} + C \\ &= \frac{\sqrt{x^2-1}}{x} + C.\end{aligned}$$

Bonus (a) **Use integration by parts twice to evaluate the *indefinite* integral**

$$\int \boldsymbol{e^{-x}} \ \mathbf{sin}\, \boldsymbol{x}\ \boldsymbol{dx}.$$

Integrate by parts with $u = e^{-x}$, $du = -e^{-x}\,dx$, $dv = \sin x\,dx$, $v = -\cos x$ to get

$$\int e^{-x} \ \sin x\ dx \ = \ -e^{-x}\cos x - \int e^{-x} \ \cos x.$$

To evaluate the resulting integral, use integration by parts again, this time setting $u = e^{-x}$, $du = -e^{-x}\,dx$, $dv = \cos x\,dx$, $v = \sin x$, to obtain

$$\int e^{-x} \ \cos x \ = \ e^{-x}\sin x + \int e^{-x}\sin x\ dx.$$

Combining these calculations, and letting I denote the original integral, we obtain

$$\begin{aligned} I &= \int e^{-x} \ \sin x\ dx \\ &= -e^{-x}\cos x - \int e^{-x} \ \cos x \\ &= -e^{-x}\cos x - \left(e^{-x}\sin x + \int e^{-x}\sin x\ dx\right) \\ &= -e^{-x}\cos x - e^{-x}\sin x - I. \end{aligned}$$

Therefore $2I = -e^{-x}\cos x - e^{-x}\sin x$ (plus a constant), that is,

$$I \ = \ \frac{-e^{-x}\cos x - e^{-x}\sin x}{2} + C.$$

Bonus (b) **Use your answer to (a) to evaluate the *improper* integral**

$$\int_0^{\infty} \boldsymbol{e^{-x}} \ \mathbf{sin}\, \boldsymbol{x}\ \boldsymbol{dx}.$$

$$\begin{aligned} \int_0^{\infty} e^{-x} \ \sin x\ dx &= \lim_{n\to\infty}\left[\int_0^{n} e^{-x} \ \sin x\ dx\right] \\ &= \lim_{n\to\infty}\left[\left.\frac{-e^{-x}\cos x - e^{-x}\sin x}{2}\right]_0^n\right] \\ &= \lim_{n\to\infty}\left[\frac{-e^{-n}\cos n - e^{-n}\sin n}{2} - \frac{-1-0}{2}\right] \\ &= \boxed{\frac{1}{2}.} \end{aligned}$$