Note: In solving Problem #1 (and possibly other problems), you may use the formulas

$$\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}} \quad \text{and} \quad \boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$$

Problem #1 [4 pts] Find the exact value of

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(-2+\frac{6i}{n}\right)^2 - 3\left(-2+\frac{6i}{n}\right) + 2 \right] \left(\frac{6}{n}\right). \qquad (\bigstar)$$

First, expand the summand to get

$$\begin{aligned} &\lim_{n\to\infty} \sum_{i=1}^{n} \left[4 - \frac{24i}{n} + \frac{36i^2}{n^2} + 6 - \frac{18i}{n} + 2 \right] \left(\frac{6}{n}\right) \\ = &\lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{72}{n} - \frac{252i}{n^2} + \frac{216i^2}{n^3} \right]. \end{aligned}$$

Now break up the sum into pieces as

$$\begin{aligned} &\lim_{n\to\infty} \left[\left(\sum_{i=1}^{n} \frac{72}{n} \right) - \left(\sum_{i=1}^{n} \frac{252i}{n^2} \right) + \left(\sum_{i=1}^{n} \frac{216i^2}{n^3} \right) \right] \\ = &\lim_{n\to\infty} \left[\frac{72}{n} \left(\sum_{i=1}^{n} 1 \right) - \frac{252}{n^2} \left(\sum_{i=1}^{n} i \right) + \frac{216}{n^3} \left(\sum_{i=1}^{n} i^2 \right) \right]. \end{aligned}$$

Use the sum formulas shown in the boxes at the top of the page (and the easier fact that $\sum_{i=1}^{n} 1 = n$) to obtain

$$\begin{aligned} &\lim_{n\to\infty} \left[72 - \frac{252}{n^2} \cdot \frac{n(n+1)}{2} + \frac{216}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \right] \\ = &\lim_{n\to\infty} \left[72 - 126\frac{n+1}{n} + 36\frac{(n+1)(2n+1)}{n^2} \right]. \end{aligned}$$

By the rules for taking limits of rational functions, this equals $72 - 126 + 36 \cdot 2 = 18$. Alternatively, you can factor this last expression completely as

$$\lim_{n\to\infty} \frac{18(n^2-n+2)}{n^2},$$

then use L'Hôpital's rule twice to evaluate the limit; you'll still get 18.

Problem #2 [3 pts] Express the limit (★) as a definite integral.

The expression certainly looks like a Riemann sum. If the integral from which it came is

$$\int_a^b f(x)\,dx,$$

then apparently $\Delta x = 6/n = (b-a)/n$, so $b - a = 6$. The right-hand sum for this integral would be $x_i^* = a + i\Delta x = -2 + 6i/n$, so $a = -2$, which means $b = a + 6 = 4$. Finally, the function f would have to be $f(x) = x^2 - 3x + 2$. Putting it all together, the answer is

$$\boxed{\int_{-2}^{4} (x^2 - 3x + 2)\,dx.}$$

Alternatively, you can regard the Riemann sum as coming from the function $g(x) = (x-2)^2 - 3(x-2) + 2$. In this case the limits of integration would be 0 and 6, so the integral is

$$\int_0^6 ((x-2)^2 - 3(x-2) + 2)\,dx.$$

Problem #3 [3 pts] Use the Fundamental Theorem of Calculus to evaluate the integral you wrote down in Problem #2.

The integral in the box is

$$\left.\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_{-2}^{6} = \left(\frac{216}{3} - \frac{108}{2} + 12\right) - \left(\frac{-8}{3} - \frac{12}{2} - 4\right) = \boxed{\frac{128}{3}.}$$

Problem #4. A particle moves along a straight line. Its velocity at time t is $v(t) = 2t - 3$.

#4a [5 pts] Find the *displacement* (i.e., change in distance) of the particle between $t = 0$ and $t = 5$. (You may use the Fundamental Theorem of Calculus.)

The displacement is

$$\begin{aligned}\int_0^5 v(t)\,dt &= \int_0^5 (2t-3)\,dt \\ &= \left. t^2 - 3t\right]_0^5 \\ &= (25 - 15) - (0 - 0) = \boxed{10.}\end{aligned}$$

#4b [5 pts] Find the *total distance traveled* between $t = 0$ and $t = 5$.

The total distance traveled is $\int_0^5 |v(t)|\,dt$. To evaluate this, we need to know where $v(t)$ is positive and where it is negative—it's not too hard to see that $v(t) \le 0$ when $t \le \frac{3}{2}$ and $v(t) \ge 0$ when $t \ge \frac{3}{2}$. Accordingly,

$$\begin{aligned}\int_0^5 |v(t)|\,dt &= \int_0^{\frac{3}{2}} (-2t+3)\,dt + \int_{\frac{3}{2}}^5 (2t-3)\,dt \\ &= \left. -t^2 + 3t\right]_0^{\frac{3}{2}} + \left. t^2 - 3t\right]_{\frac{3}{2}}^5 \\ &= \left(-\frac{9}{4} + \frac{9}{2}\right) + \left((25 - 15) - \left(\frac{9}{4} - \frac{9}{2}\right)\right) = \boxed{\frac{29}{2}.}\end{aligned}$$

Note that $|2t-3| \neq 2t+3$; absolute values don't work that way. (Try sketching the graphs of these two functions; you'll see that they are not the same.)

Bonus Problems [3 honors points each] Evaluate each of the following integrals, *without using the Fundamental Theorem of Calculus.* Your answer should be exact, not approximate.

(a) $\displaystyle \int_2^4 \boldsymbol{xe^{\cos x}\,dx} \;+\; \int_4^{-2} \boldsymbol{xe^{\cos x}\,dx}$

Using Rule #5 (p. 362), this equals

$$\int_2^{-2} xe^{\cos x}\,dx$$

and by the unnumbered rule in the middle of p. 361, this in turn becomes

$$\int_{-2}^{2} -xe^{\cos x}\,dx. \qquad (\clubsuit)$$

Let $f(x) = -xe^{\cos x}$. Then $f(-x) = xe^{\cos(-x)} = xe^{\cos x} = -f(x)$; that is, $f(x)$ is an **odd** function. It follows that the integral ($\clubsuit$) is $\boxed{\text{zero.}}$ (This fact is mentioned explicitly as equation (6) on p. 391, but it makes sense in terms of areas—we could break up the integral ($\clubsuit$) as

$$\int_{-2}^{0} f(x)\,dx \;+\; \int_0^2 f(x)\,dx. \qquad (\diamondsuit)$$

Because $f(x)$ is odd, the region above the graph and below the x-axis between $x=-2$ and $x=0$ is congruent—to the area below the graph and above the x-axis between $x=0$ and $x=2$. If the area of this region is A, then the integral ($\diamondsuit$) equals $-A+A=0$.

(b) $\displaystyle \int_0^{\pi/3} \boldsymbol{(2-\sin^2 x)\,dx} \;-\; \int_{-\pi}^{-2\pi/3} \boldsymbol{\cos^2 x\,dx}$

Here we first need to use the fact that $\cos x = -\cos(x+\pi)$ (this comes from the periodicity of the cosine function), so $\cos^2 x = \cos^2(x+\pi)$. SO we can change the limits on the second integral to 0 and $\pi/3$ (instead of $-\pi$ and $-2\pi/3$), then use the rules of §5.2 (pp. 361–363):

$$\begin{aligned}
\int_0^{\pi/3} (2-\sin^2 x)\,dx \;-\; \int_{-\pi}^{-2\pi/3} \cos^2 x\,dx &= \int_0^{\pi/3} (2-\sin^2 x)\,dx \;-\; \int_0^{\pi/3} \cos^2 x\,dx \\
&= \int_0^{\pi/3} (2-\sin^2 x - \cos^2 x)\,dx \\
&= \int_0^{\pi/3} 1\,dx \\
&= \boxed{\pi/3.}
\end{aligned}$$