Problem #1. Let $C(x) = x^{3/2} - 3\sqrt{x}$.

#1a. [3 pts] Find the linearization of $C(x)$ at the point $(9, 18)$.

The linearization is the equation $L(x) = C'(3)(x-9) + 18$. We have $C'(x) = \frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2}$, so $C'(9) = \frac{9}{2} - \frac{1}{2} = 4$, so the linearization is $L(x) = 4(x-9) + 18 = 4x - 18$.

#1b. [3 pts] Use your answer to Problem 1a to estimate the value of $C(10)$.

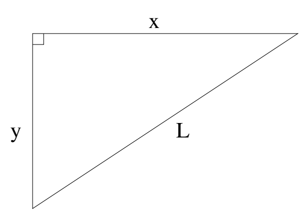
#1a gives $C(10) \approx L(10) = 22$.

#1c. [2 pts] Use the function $C''(x)$ to determine whether your estimate in Problem 1b was higher or lower than the actual value of $C(10)$. (You can calculate $f(10)$ independently if you want to, but to receive full credit, your answer must make use of C''.)

$C''(x) = \frac{3}{4}x^{-1/2} + \frac{3}{4}x^{-3/2}$. Whatever the value of $C''(10)$, it is definitely positive (indeed, $C''(x) > 0$ for all positive x), so the graph of C is concave up near $x = 10$. That indicates that the approximation in #1b is an **underestimate**, because the tangent line at $(9, 18)$ — that is, the graph of $L(x)$ — lies below the graph of C. (Indeed, $C(10) = 22.13594\cdots > L(10)$.)

Problem #2. [6 pts] Two robbers are fleeing the scene of a crime they have just committed. Precisely at 6:00 PM, Robber #1 climbs into a stolen school bus and drives due south at a constant rate of 60 mph. Precisely at 6:10 PM, Robber #2, who has stayed behind to destroy incriminating evidence, hops on her motorized tricycle and zooms away due east at a constant speed of 120 mph. How fast is the distance between the two robbers changing at 6:30 PM?

Let y be the distance that Robber #1 has traveled, and let x be the distance that Robber #2 has traveled. Then $L = \sqrt{x^2 + y^2}$ is the distance between the two robbers, as in the drawing below.



To find $L' = \frac{dL}{dt}$, we implicitly differentiate the equation $L^2 = x^2 + y^2$ with respect to t, obtaining $2LL' = 2xx' + 2yy'$, and solving for L' gives

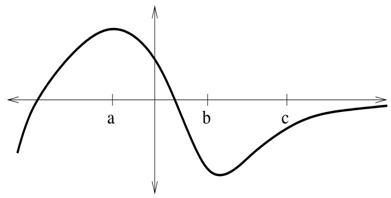
$$L' = \frac{xx' + yy'}{L}. \tag{1}$$

(Alternatively, we could differentiate $L = \sqrt{x^2 + y^2}$ with respect to t, but it is easier to get rid of the square root sign before differentiating.)

We are given that $x' = 120$ mph and $y' = 60$ mph. At 6:30 PM, Robber #1 has traveled 30 miles, and Robber #2 has gone 40 miles. So $y = 30$, $x = 40$, and $L = \sqrt{30^2 + 40^2} = 50$. Plugging all these values into (1) gives

$$L' = \frac{120(40) + 60(30)}{50} = 132 \text{ mph.}$$

Problem #3. Let f be the function whose graph is shown below.



Explain what is likely to happen if Newton's method is used to find a root of f with each of these possible initial estimates for r_0:

#3a. [2 pts] $r_0 = a$

This appears to be a critical point of $f(x)$—that is $f'(a) = 0$. So Newton's method will break down at the very first interation, since

$$r_1 = r_0 - \frac{f(r_0)}{f'(r_0)} = r_0 - \frac{f(a)}{f'(a)}$$

is undefined.

#3b. [2 pts] $r_0 = b$

Here Newton's method is probably going to converge to the root of f in the interval $(0, b)$.

#3c. [2 pts] $r_0 = c$

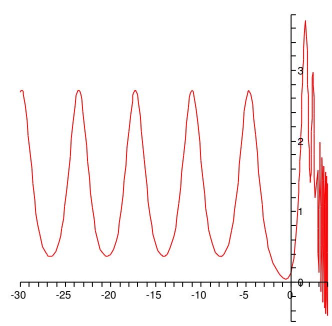
Here the values $r_1, r_2, \ldots$ are likely to increase without bound, because of the horizontal asymptote of the graph of f.

Bonus problem [4 honors pts] Use Newton's Method to find the smallest real root of

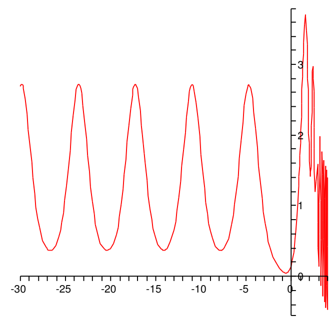
$$K(x) = e^{\sin x} - \sin(e^x).$$

Your answer should be accurate to four decimal places.

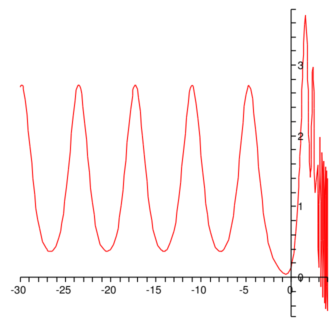
Here's the graph of $K(x)$ for values of x in the interval $[-30, 5]$:



Apparently the smallest zero lies in $[0, 5]$. Zooming in to that interval...



...we can see that the root is pretty close to 3. Zooming even further in to $[3,4]$...



...we might choose $r_0 = 3.2$ as our first approximation. Now we can apply Newton's Method. Since

$$K'(x) \;=\; (\cos x)e^{\sin x} - (\cos e^x)e^x,$$

we can calculate

$$r_1 = r_0 - \frac{K(r_0)}{K'(r_0)} \approx 3.27118, \qquad r_2 = r_1 - \frac{K(r_1)}{K'(r_1)} \approx 3.26561,$$

$$r_3 = r_2 - \frac{K(r_2)}{K'(r_2)} \approx 3.26632, \qquad r_4 = r_3 - \frac{K(r_3)}{K'(r_3)} \approx 3.26633,$$

$$r_5 = r_4 - \frac{K(r_4)}{K'(r_4)} \approx 3.26633.$$

Apparently the root (to four decimal places) is 3.2663.