The entire quiz (except the two bonus problems) is about the function

$$\boldsymbol{f(x) = \frac{x^2}{x-2}}.$$

#1. [2 pts] Find all critical points of f.

Start by calculating $f'(x)$, using the Quotient Rule:

$$f'(x) = \frac{(x-2)(2x) - (x^2)(1)}{(x-2)^2} = \frac{x^2 - 4x}{(x-2)^2} = \frac{x(x-4)}{(x-2)^2}.$$

So the critical points are at $x = 2$ (where $f'(x)$ is undefined) and $x = 0$, $x = 4$ (where $f'(x) = 0$).

#2. [3 pts] Find all intervals on which f is increasing, and all intervals on which f is decreasing.

By #1, the direction of f is constant on each of the intervals $(-\infty, 0)$, $(0, 2)$, $(2, 4)$, and $(4, \infty)$. Pick a sample point in each of these intervals and determine the sign of f':

Interval	Sample point	Value of f' at sample point	Direction of f
$(-\infty, 0)$	-1	$5/9$	Increasing
$(0, 2)$	1	-3	Decreasing
$(2, 4)$	3	-3	Decreasing
$(4, \infty)$	1000	≈ 1	Increasing

Some of you said that $f(x)$ was decreasing on the interval $(0, 4)$ [rather than on the two separate intervals $(0, 2)$ and $(2, 4)$]. This isn't correct: for example, $f(-1) = -1$ but $f(3) = 9$. This is why a vertical asymptote really must be considered as a critical point!

#3. [2 pts] For each critical point that you found, determine whether it is a local minimum, a local maximum, or neither.

Using the First Derivative Test and the table in #2, we see that $x = 0$ is a local maximum, $x = 4$ is a local minimum, and $x = 2$ is neither (indeed, it's not even in the domain of f).

#4. [3 pts] Find the absolute maximum and absolute minimum of $f(x)$ on the interval $[5, 8]$.

This interval is a subset of $(4, \infty)$, on which f is increasing. So the absolute minimum occurs at $x = 5$ and the absolute maximum occurs at $x = 8$.

(I had meant to ask about the interval $[3, 8]$, but apparently a typo crept in, making the problem easier than I hazd intended....)

#5. [2 pts] Find all inflection points of f.

We'll need the second derivative:

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left(\frac{x^2-4x}{(x-2)^2}\right) \\ &= \frac{(x-2)^2(2x-4)-(x^2-4x)(2(x-2))}{(x-2)^4} \\ &= \frac{(x-2)(2x-4)-(x^2-4x)(2)}{(x-2)^3} \\ &= \frac{2x^2-4x-4x+8-2x^2+8x}{(x-2)^3} \\ &= \frac{8}{(x-2)^3}. \end{aligned}$$

So $x = 2$ is the only inflection point.

#6. [3 pts] Find all intervals on which f is concave up, and all intervals on which f is concave down.

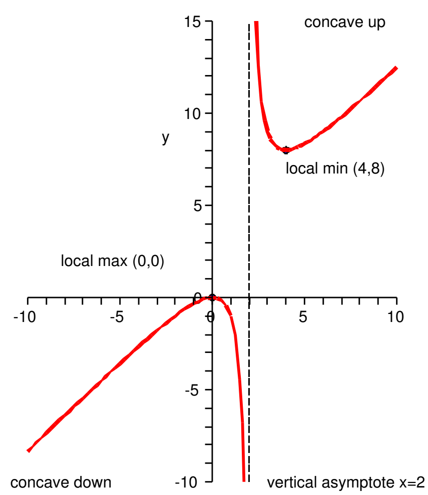
From the calculation in #5, we see that $f''(x) > 0$ when $x > 2$ and $f''(x) < 0$ when $x < 2$. Therefore, f is concave down on $(-\infty, 2)$ and concave down. on $(2, \infty)$.

#7. [2 pts] Find all vertical asymptotes of f, and describe the behavior of $f(x)$ as $x \to \infty$ or $x \to -\infty$.

There is a single vertical asymptote at $x = 2$, and $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

#8. [3 pts] Sketch the graph of $f(x)$.

Here is the precise graph of $f(x)$, produced using Maple. Your graph should strongly resemble it.



Bonus problem #1 [4 pts] Let $f(x)$ and $g(x)$ be differentiable functions of x. Find a formula for

$$\frac{d}{dx}\left[f(x)^{g(x)}\right],$$

and explain why your formula works both for power functions (where $f(x) = x$ and $g(x)$ is a constant) and exponential functions (where $f(x)$ is a constant and $g(x) = x$).

Abbreviate $f = f(x)$ and $g = g(x)$. Let $y = f^g$, and use logarithmic differentiation:

$$\ln y = \ln f^g = g \cdot \ln f$$

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}[g \cdot \ln f]$$

$$\frac{y'}{y} = g' \ln f + \frac{gf'}{f}$$

so

$$y' = y\left(g' \ln f + \frac{gf'}{f}\right) = f^g\left(g' \ln f + \frac{gf'}{f}\right). \qquad (*)$$

If $f(x) = x$ and $g(x) = c$ is a constant, then $f' = 1$ and $g' = 0$, so (*) becomes

$$y' = x^c \cdot \frac{c}{x} = cx^{c-1},$$

which confirms the Power Rule.

If $g(x) = x$ and $f(x) = c$ is a constant, then $g' = 1$ and $f' = 0$, so (*) becomes

$$y' = c^x(\ln c),$$

which confirms our rule for differentiating exponential functions.

Bonus problem #2 [4 pts] Recall the statement of the Mean Value Theorem: for every function $f(x)$ that is differentiable on a closed interval $[a, b]$, there is at least one number c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \qquad (*)$$

Show that the theorem becomes false if "differentiable" is replaced with "continuous". (That is, you need to come up with a function f and a closed interval $I = [a, b]$ such that f is continuous on I, but equation $(*)$ is false for *every* number c in I.)

There are lots of possibilities. For example, let $I = [a, b] = [-1, 1]$, and let $f(x) = |x|$, which is continuous but not differentiable on I. Then

$$\frac{f(b) - f(a)}{b - a} = \frac{1 - 1}{1 - (-1)} = 0,$$

but there is no number c in I such that $f'(c) = 0$. (Recall that $f'(x) = 1$ for x positive; $f'(x) = -1$ for x negative; and $f'(0)$ does not exist.)