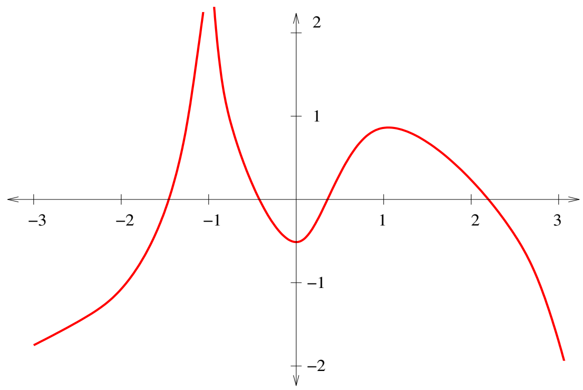
Problem #1 refers to the function $f(x)$ given by the following graph.



#1a. Where is f differentiable?

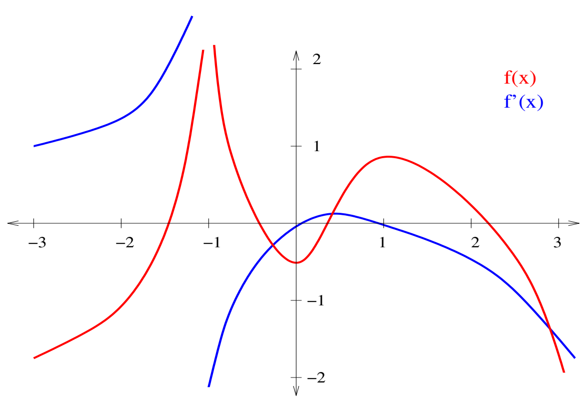
f is differentiable everywhere except at $x = -1$, where there is a discontinuity.

#1b. Sketch the graph of $f'(x)$. Your graph doesn't have to be to scale, but you should clearly label all interesting points (such as zeroes and local max/min points).

Since we don't have a formula for $f(x)$, we can't be sure about the exact graph of $f'(x)$, but some elements that must be present are:

- $f'(x) > 0$ on $(-3, -1)$ and on $(0, 1)$ (where f is increasing)
- $f'(x) < 0$ on $(-1, 0)$ and on $(1, 3)$ (where f is decreasing)
- $f'(x) = 0$ at $x = 0$ and $x = 1$ (respectively a local minimum and a local maximum of f
- $f'(-1)$ is undefined (where f' is not continuous, hence not differentiable)
- $\lim_{x \to -1^-} f'(x) = \infty$ and $\lim_{x \to -1^+} f'(x) = -\infty$
- f' has a local maximum somewhere in the interval $(0, 1)$. It's hard to say exactly where, but the local maximum of f' corresponds to an inflection point of f. In fact, as long as $f'(0) = f'(1) = 0$ and f' is positive on $(0, 1)$, there *must* be a local maximum somewhere on $(0, 1)$ (as long as you draw the graph of f' continuously, which you should).

The graphs of $f(x)$ and $f'(x)$ should look something like this:

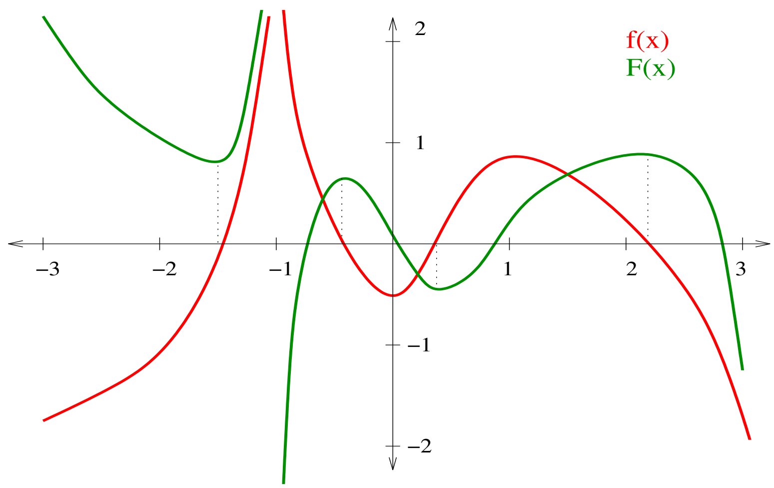


#1c. [5 pts] Sketch the graph of an antiderivative of $f(x)$.

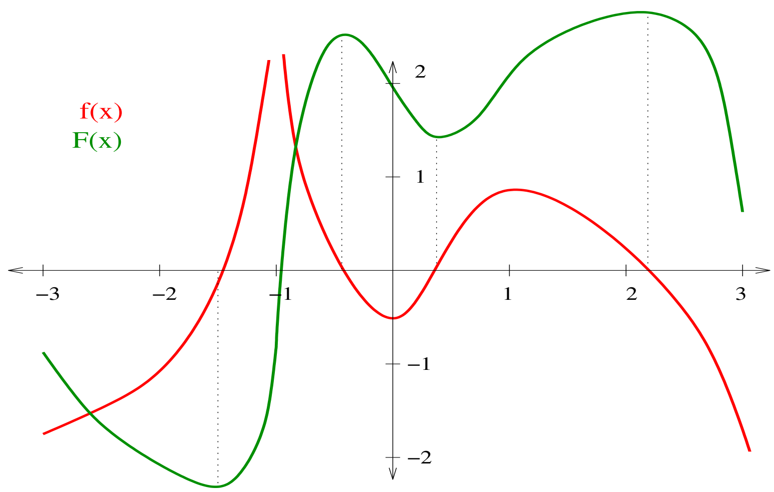
Again, we can't be sure about the exact graph of the antiderivative (let's call it $F(x)$) but it must certainly satisfy the following properties:

- The zeroes of f, which are roughly at $-3/2$, $-1/2$, $1/3$ and $9/4$, are local extrema of F.
- F is decreasing where f is negative, that is, on $(-\infty, -3/2) \cup (-1/2, 1/3) \cup (9/4, \infty)$.
- F is increasing where f is positive, that is, on $(-3/2, -1) \cup (-1, -1/2) \cup (1/3, 9/4)$.
- F has an inflection point when f has a local minimum or maximum, namely at $x = 0$ and $x = 1$.
- F is concave up on $(-3, -1) \cup (0, 1)$, where f is increasing, and concave down on $(-1, 0) \cup (1, \infty)$, where f is decreasing.

The graphs of $f(x)$ and its antiderivative $F(x)$ should look something like this. (I've marked the zeroes of f, which correspond to local extrema of F.)



As one person noticed, F might be continuous at $x = -1$ even though f isn't. If so, then F must have a vertical tangent line:



Problem #2 refers to the function

$$\boldsymbol{p(x) = \frac{3}{8}x^4 - \frac{5}{2}x^3 + 6x^2 - 9x + 15.}$$

#2a. Find $p'(x)$.

Use the Souped-Up Power Rule, $\frac{d}{dx}cx^n = cnx^{n-1}$, and the Addition/Subtraction Rules, $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$:

$$\begin{aligned} p'(x) &= \frac{3}{8}(4x^3) - \frac{5}{2}(3x^2) + 6(2x) - 9(1) + 0 \\ &= \frac{3}{2}x^3 - \frac{15}{2}x^2 + 12x - 9. \end{aligned}$$

#2b. Find $p'(x)$.

$$\begin{aligned} p''(x) \;=\; \frac{d}{dx}(p'(x)) &= \frac{3}{2}(3x^2) - \frac{15}{2}(2x) + 12(1) - 9(0) \\ &= \frac{9}{2}x^2 - 15x + 12. \end{aligned}$$

#2c. Find all inflection points of $p(x)$.

The inflection points of $p(x)$ are the roots of $p''(x) = \frac{9}{2}x^2 - 15x + 12 = \frac{3}{2}(3x-4)(x-2)$, namely $x = \frac{4}{3}$ and $x = 2$. (These values can also be found using the quadratic formula.)

#2d. [2 pts] What is the **smallest** value of n such that the n^{th} derivative of p is zero?

p is a polynomial of degree 4, so its fourth derivative is a constant and its fifth derivative is zero (as are all subsequent ones). The answer is $n = 5$.

Bonus problem: Let $g(x)$ be any function. Use the Product Rule to find formulas for

$$\frac{d}{dx}\left[g(x)^2\right], \quad \frac{d}{dx}\left[g(x)^3\right], \quad \textbf{and} \quad \frac{d}{dx}\left[g(x)^4\right]$$

in terms of $g(x)$ and $g'(x)$. Based on your answers, can you make a more general conjecture about derivatives of powers of a function?

I'll abbreviate $g(x)$ and $g'(x)$ by g and g' respectively to keep the notation a bit less bulky.

$$\begin{aligned}\frac{d}{dx}\left(g^2\right) &= \frac{d}{dx}\left(g \cdot g\right) \\ &= gg' + g'g \;=\; 2gg';\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}\left(g^3\right) &= \frac{d}{dx}\left(g \cdot g^2\right) \\ &= g \cdot \frac{d}{dx}\left(g^2\right) + g' \cdot g^2 \\ &= g \cdot 2gg' + g'g^2 \\ &= 3g^2g';\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}\left(g^4\right) &= \frac{d}{dx}\left(g^2 \cdot g^2\right) \\ &= g^2 \cdot \frac{d}{dx}\left(g^2\right) + \big(dx(g^2)\big) \cdot g^2 \\ &= (g^2)(2gg') + (2gg')(g^2) \\ &= 4g^3g'.\end{aligned}$$

The pattern seems to be that

$$\frac{d}{dx}\left[g(x)^n\right] \;=\; n \cdot g(x)^{n-1} \cdot g'(x),$$

for all positive integers n. In fact, this is true; we'll be able to prove it using the Chain Rule.