#1. Determine all horizontal and vertical asymptotes of the function $f(x) = \dfrac{-3x^2 - 2x + 6}{x^2 + 3x - 4}$.

Vertical asymptotes of $f(x)$ occur at the zeroes of its denominator $x^2 + 3x - 4 = (x-1)(x+4)$, that is, at $\boxed{x = 1}$ and $\boxed{x = -4.}$

To find the horizontal asymptotes, evaluate the limits

$$\lim_{x\to\infty} f(x) \quad \text{and} \quad \lim_{x\to-\infty} f(x).$$

Since the numerator and denominator of $f(x)$ are rational functions of the same degree (namely 2), both these limits equal the ratio of the leading coefficients, that is, $-3/1 = 3$. So $f(x)$ has one horizontal asymptote, naamely $\boxed{y = -3.}$ By the way, these limits can also be calculated algebraically:

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{-3x^2 - 2x + 6}{x^2 + 3x - 4} \\
&= \lim_{x\to\infty} \left(\frac{-3x^2 - 2x + 6}{x^2 + 3x - 4}\right)\left(\frac{x^{-2}}{x^{-2}}\right) \\
&= \lim_{x\to\infty} \left(\frac{-3 - 2x^{-1} + 6x^{-2}}{1 + 3x^{-1} - 4x^{-2}}\right) \\
&= \frac{\lim_{x\to\infty}(-3) - \lim_{x\to\infty}(2x^{-1}) + \lim_{x\to\infty}(6x^{-2})}{\lim_{x\to\infty}(1) + \lim_{x\to\infty}(3x^{-1}) - \lim_{x\to\infty}(4x^{-2})} \;=\; \frac{-3 - 0 + 0}{1 + 0 - 0} \;=\; -3.
\end{aligned}$$

Notice that "$\lim_{x\to\infty}$" can be replaced with "$\lim_{x\to-\infty}$" throughout without changing any of the algebra.

#2. Evaluate $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{2x}$.

When $x < 0$ (therefore, as $x \to -\infty$), $1/x = x^{-1} = -\sqrt{x^{-2}}$. Therefore:

$$\begin{aligned}
\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{2x} &= \lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{2x} \cdot \frac{x^{-1}}{x^{-1}} \\
&= \lim_{x\to-\infty} \frac{-\sqrt{1 + x^{-2}}}{2} = \boxed{-\frac{1}{2}}
\end{aligned}$$

- Many of you started by writing down the equation

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{2x} = \lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{2x} \cdot \frac{\sqrt{x^2-1}}{\sqrt{x^2-1}},$$

which is algebraically correct, but doesn't really help—when you multiply out the numerator, there will still be a radical! (This is not quite the same thing as rationalizing the numerator of an expression by using its conjugate (see, e.g., Example 6 on p. 112 of the textbook), because here the addition occurs *underneath* the radical.)

• Another common mistake was to look at the wrong function! Some of you claimed the limit was $-\infty$, and gave as justification either a table of values or a graph of the function

$$\frac{\sqrt{x^2+1}}{2}x \qquad \text{rather than the correct} \qquad \frac{\sqrt{x^2+1}}{2x}.$$

What you probably did was enter something like the following into your calculator:

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√‾  (  x  ∧  2  +  1  )  /  2  *  x.
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This will give the first (incorrect) expression. The denominator needs to be enclosed in parentheses:

```
√‾  (  x  ∧  2  +  1  )  /  (  2  *  x  )
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will give the correct expression.

#3. Evaluate $\lim\limits_{x\to 0^-} \dfrac{\sqrt{x^2+1}}{2x}$.

As $x \to 0$ from the left, the numerator $\sqrt{x^2+1}$ approaches 1, while the denominator $2x$ approaches zero through negative values. It follows that the given limit is $\boxed{-\infty.}$

Another possibility is to calculate

$$\lim_{x\to 0} \frac{\sqrt{x^2+1}}{2x} = \lim_{x\to 0} \frac{\sqrt{x^2+1}\cdot x^{-1}}{2x\cdot x^{-1}}$$

$$= \lim_{x\to 0} \frac{-\sqrt{1+x^{-2}}}{2}$$

and then to notice that the numerator of this expression approaches $-\infty$.

#4. Let $g(x) = \dfrac{1}{\sqrt{x+1}}$. Find the slope of the tangent line to the graph of g at $(0,1)$.

We want to calculate $g'(0)$. There are two ways to do this (depending on which definition of derivative we choose), but they will look approximately the same. Here's one way:

$$\begin{aligned}
g'(0) &= \lim_{h\to 0} \frac{g(0+h)-g(0)}{h} \quad = \quad \lim_{h\to 0} \frac{\left(\frac{1}{\sqrt{h+1}} - 1\right)}{h} \\
&= \lim_{h\to 0} \frac{\left(\frac{1}{\sqrt{h+1}} - \frac{\sqrt{h+1}}{\sqrt{h+1}}\right)}{h} \\
&= \lim_{h\to 0} \left(\frac{1-\sqrt{h+1}}{h\sqrt{h+1}}\right) \\
&= \lim_{h\to 0} \left(\frac{1-\sqrt{h+1}}{h\sqrt{h+1}} \cdot \frac{1+\sqrt{h+1}}{1+\sqrt{h+1}}\right)
\end{aligned}$$

(this is the "dirty trick" step)

$$= \lim_{h\to 0}\left(\frac{1-(h+1)}{h\sqrt{h+1}(1+\sqrt{h+1})}\right)$$
$$= \lim_{h\to 0}\left(\frac{-h}{h\sqrt{h+1}(1+\sqrt{h+1})}\right)$$
$$= \lim_{h\to 0}\left(\frac{-1}{\sqrt{h+1}(1+\sqrt{h+1})}\right)$$

(the cancellation we have been hoping for; now we can plug in $h = 0$)

$$= \frac{-1}{\sqrt{1}(1+\sqrt{1})} \;=\; \boxed{\frac{1}{2}}$$

#5. Let $b(x) = 2x^2 - 5x + 6$. Find a formula for the derivative $b'(x)$.

$$b'(x) = \lim_{h\to 0}\frac{b(x+h)-b(x)}{h} \;=\; \lim_{h\to 0}\frac{\big[(x+h)^2 - 5(x+h) + 6\big] - \big[x^2 - 5x + 6\big]}{h}$$

$$= \lim_{h\to 0}\frac{\big[x^2 + 2xh + h^2 - 5x - 5h + 6\big] - \big[x^2 - 5x + 6\big]}{h}$$
$$= \lim_{h\to 0}\frac{2xh + h^2 - 5h}{h}$$
$$= \lim_{h\to 0} 2x + h - 5 \;=\; \boxed{2x - 5}$$

Bonus problem Suppose that $f(x)$ is an even function (that is, $f(x) = f(-x)$ for all x). It is a fact that the derivative $f'(x)$ is an odd function (that is, $f'(x) = -f'(-x)$ for all x).

#E1. Prove this fact algebraically.

We'll use the definition of the derivative and the condition that f is even to prove that f' is odd:

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$
$$= \lim_{x\to a}\frac{f(-x)-f(-a)}{(-a)-(-x)}$$
$$= -\lim_{x\to a}\frac{f(-x)-f(-a)}{(-x)-(-a)}.$$

Define $w = -x$ and $b = -a$, so that $w \to b$ as $x \to a$, and the last expression becomes

$$-\lim_{w\to b}\frac{f(w)-f(b)}{w-b},$$

which is precisely the definition of $f'(w)$ (with some letters changed).

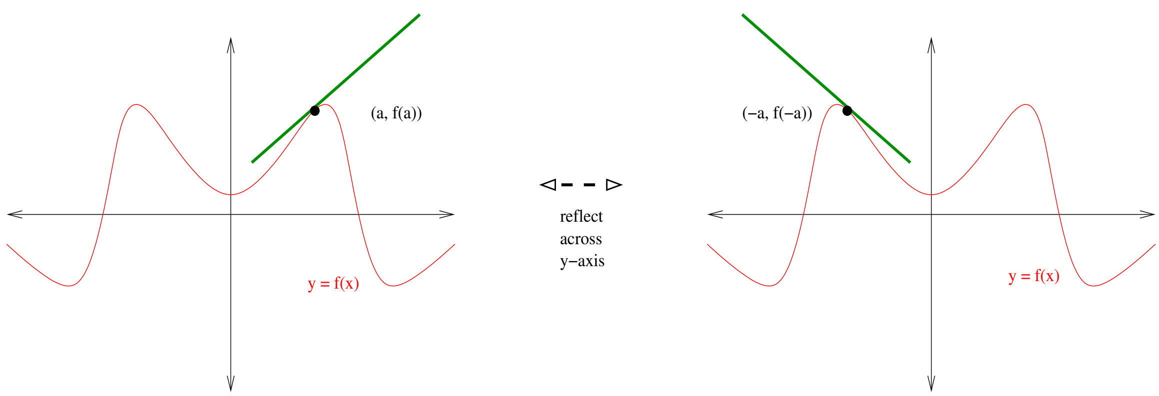
- Some of you were confused by the terms "even function" and "odd function." These don't refer to the degree of a polynomial, but to the symmetries given by the equations $f(x) = f(-x)$ (even) and $f'(x) = -f'(-x)$ (odd). Graphically, the graph of an even function is symmetric with respect to reflection across the y-axis, while the graph of an odd function is symmetric with respect to rotating 180° around the origin. (A function doesn't have to be a polynomial to be even or odd. Yes, if n is even then $f(x) = x^n$ is an even function, and if n is odd then $g(x) = x^n$ is an odd function. But the terminology is more general: for example, $h(x) = \cos x$ is also even.)

- Some of you answered this question by picking one particular even function, often $f(x) = x^2$, then showing that its derivative (in this case, $f'(x) = 2x$) was odd. While correct in itself, that doesn't tell us anything about all the other even functions in the world. The principle here is that **you can't prove a general fact by working it out for one example.** The reason that the argument above qualifies as a proof is that it doesn't use anything about $f(x)$ other than the property that it is even; therefore, it is valid for every even function you can dream up.

#E2. Explain this fact in terms of the graphs of the functions $f(x)$ and $f'(x)$.

If $f(x)$ is even, then we can reflect its graph across the y-axis without changing the graph. Suppose that we draw the tangent line to the graph at $(a, f(a))$ and reflect this line along with the graph. Then the reflected line will be the tangent line at $(-a, f(a)) = (-a, f(-a))$, but its slope will have been multipled by -1 as a result of the reflection. Therefore $f'(-a) = -f'(a)$. This works for any value a in the domain of f, so we can conclude that f' is an odd function.

For example, the graph and tangent line might look like this:



(Yes, the figure only shows what's going on for one specific even function. But the argument just given applies to any even function; the figure is just there to help make the geometric idea explicit.)