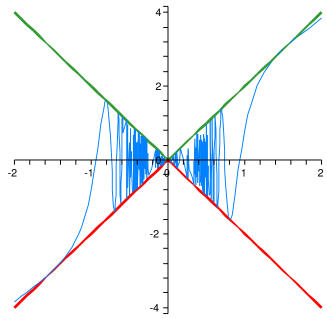
#1. [4 pts] Use the Squeeze Theorem to evaluate

$$\lim_{x\to 0} 2x\ \sin(e^{1/x^2}).$$

Since $-1 \le \sin y \le 1$ for all y (no matter whether y is x, e^{1/x^2}, or anything else, it follows that

$$-2|x| \ \le\ 2x\ \sin(e^{1/x^2})\ \le\ 2|x|$$

for all x. This is illustrated by the following figure, showing the graphs of $y = 2|x|$ (green), $y = -2|x|$ (red), and $y = 2x\ \sin(e^{1/x^2})$ (blue).



As we know algebraically (or from the picture), $\lim_{x\to 0} -2|x| = \lim_{x\to 0} 2|x| = 0$. Therefore, by the Squeeze Theorem,

$$\lim_{x\to 0} 2x\ \sin(e^{1/x^2}) = 0.$$

#2. Let $f(x) = [\![\cos\ x]\!]$. (Remember that $[\![y]\!]$ means the greatest integer less than or equal to y.)

#2a. [3 pts] Explain why $f(x)$ is not continuous at $x = 0$.

If x is slightly smaller or slightly larger than 0, then $\cos\ x$ is a number slightly less than 1, so that $[\![\cos\ x]\!] = 0$. That is,

$$\lim_{x\to 0^+} f(x)\ =\ \lim_{x\to 0^-} f(x)\ =\ \lim_{x\to 0} f(x)\ =\ 0.$$

On the other hand,

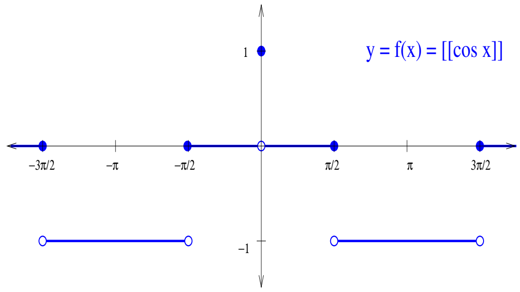
$$f(0) = [\![\cos\ 0]\!] = [\![1]\!] = 1.$$

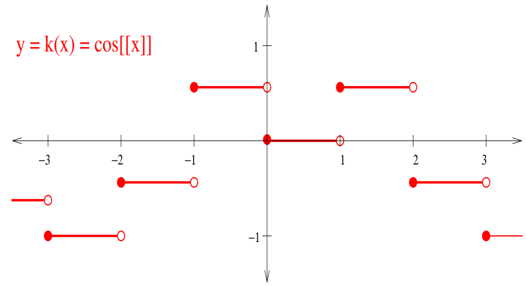
Therefore, $f(x)$ is not continuous at $x = 0$.

Note: A common mistake was to confuse $f(x)$ with the function $k(x) = \cos[\![x]\!]$, which is not the same thing. For example,

$$[\![\cos\ 0.1]\!] = [\![0.9950\ldots]\!] = 0, \quad \text{but } \cos[\![0.1]\!] = \cos 0 = 1.$$

On the other hand, it is true that $k(x)$ has a discontinuity at $x = 0$, but for a different reason: $\lim_{x\to 0^+} k(x) = 1$, while $\lim_{x\to 0^-} k(x) = cos(-1) \approx 0.540302$, so $\lim_{x\to 0} k(x)$ does not exist. Compare the graphs of $f(x)$ and $k(x)$:





#2b. [1 pt] Is the discontinuity at $x = 0$ removable, jump, or infinite?

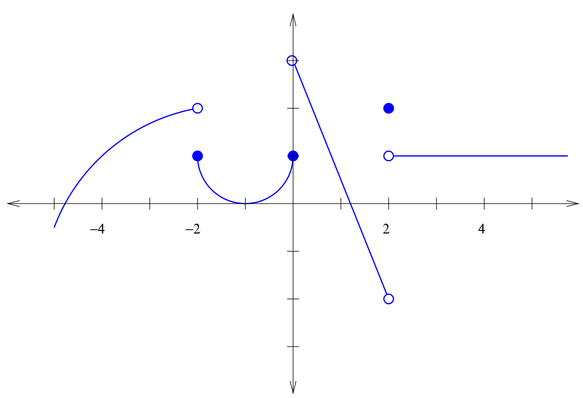
It's removable, because $\lim_{x \to 0} f(x)$ exists. The discontinuity could be removed by redefining $f(0) = 0$, which would change only a single point of the graph.

Note that $k(x)$ has a jump discontinuity (rather than a removable discontinuity) at $x = 0$, because $\lim_{x \to 0^+} k(x)$ and $\lim_{x \to 0^-} k(x)$ each exist but are not equal. If you answered #2a for the function $k(x)$, then you received credit in #2b for answering "jump" in #2b.

#3. [4 pts] Sketch the graph of a function that is

- **continuous from the right, but not from the left, at $x = -2$;**
- **continuous from the left, but not from the right, at $x = 0$;**
- **neither continuous from the left nor continuous from the right at $x = 2$; and**
- **continuous for all other values in its domain.**

There are many possible answers. Here's one. Notice which points are included and which ones are excluded.



#4. [4 pts] Find all values of c such that the function

$$g(x) = \begin{cases} 4x^2 - c^2 x + 4 & \text{for } x < -1 \\ -2cx & \text{for } x \geq -1 \end{cases}$$

is continuous on $(-\infty, \infty)$.

First of all, no matter what c is, $g(x)$ will be continuous at a for every $a \neq -1$ (because it is indistinguishable from a polynomial near $x = a$). So the only possible problem occurs at $x = -1$. We have

$$\begin{aligned} \lim_{x\to -1^-} g(x) &= \lim_{x\to -1^-} 4x^2 - c^2x + 4 &= 4(-1)^2 - c^2(-1) + 4 &= c^2 + 8, \\ \lim_{x\to -1^+} g(x) &= \lim_{x\to -1^+} -2cx &= -2c(-1) &= 2c. \end{aligned}$$

Also, we have $g(-1) = 2c$. So the values of c for which $g(x)$ is continuous at $x = -1$ are the solution(s) of the equation $c^2 + 8 = 2c$, or equivalently

$$c^2 - 2c + 8 = 0.$$

But this doesn't have any solutions— the quadratic formula gives $c = \frac{2\pm\sqrt{-28}}{2}$. So $g(x)$ is not continuous for any c.

#5. [4 pts] Let $h(x) = \dfrac{x^2+1}{x}$.

Two Math 141 students are arguing about whether the equation $h(x) = 0$ has a solution.

Student #1: **"Yes, it does. After all, $h(-1) = -2$ and $h(1) = 2$, and 0 is between -2 and 2. So the Intermediate Value Theorem says that there has to be some value a in the interval $[-1, 1]$ such that $h(a) = 0$."**

Student #2: **"No, it doesn't. If $h(x) = 0$, then $x^2 + 1$ has to be zero, and that's impossible."**

Someone has made a mistake. Decide whose reasoning is faulty, and explain what their mistake is.

Student #1 is incorrect. The Intermediate Value Theorem does not apply here, because h is not continuous on $[-1, 1]$—it has an infinite discontinuity at $x = 0$. In fact, Student #2's logic is correct.

Bonus problem [4 honors pts] Consider the function

$$k(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ x & \text{if } x \text{ is irrational.} \end{cases}$$

For what values of a (if any) is $k(x)$ continuous at $x = a$?

$k(x)$ is continuous at $x = 0$. Notice that $-|x| \leq k(x) \leq |x|$ for all x, and $\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0$. Therefore, by the Squeeze Theorem, $\lim_{x\to 0} k(x) = 0 = k(0)$.

But k is not continuous anywhere else. For instance, let $a = 1$. Then

$$k(1.1) = 0, \quad k(1.01) = 0, \quad k(1.001) = 0, \quad k(1.0001) = 0, \quad \ldots$$

so if $\lim_{x\to 1} k(x)$ exists, it can only equal 0. On the other hand,

$$k(1 + 1/e) = 1 + 1/e, \quad k(1 + 1/e^2) = 1 + 1/e^2, \quad k(1 + 1/e^3) = 1 + 1/e^3, \quad \ldots$$

so if $\lim_{x\to 1} k(x)$ exists, it can only equal 1. Putting these two observations together, we conclude that $\lim_{x\to 1} k(x)$ does not exist. A similar argument can be used to show that $k(x)$ is discontinuous at every irrational number.