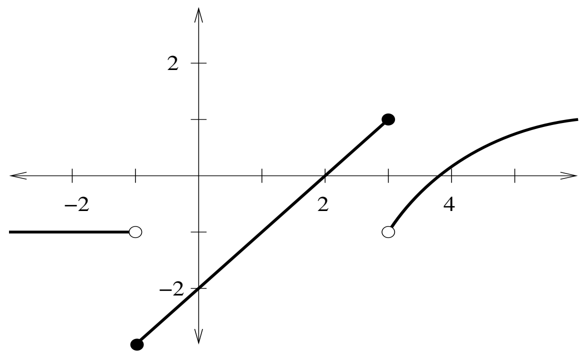
#1. [2 pts each] Let $f(x)$ be the function given by the following graph:



For each of the following limits, either evaluate it or state that it does not exist.

(a) The graph tells us that

$$\lim_{x\to 0} f(x) = \boxed{-2.}$$

(b) Using the Limit Laws:

$$\lim_{x\to 0} 3\left(f(x)^2 - \frac{2}{f(x)} + x^2 - 4\right) = 3\left(\lim_{x\to 0} f(x)\right)^2 - \frac{6}{\left(\lim_{x\to 0} f(x)\right)} + 3\left(\lim_{x\to 0} x\right)^2 - 12$$

$$= 3(-2)^2 - \frac{6}{-2} + 3(0)^2 - 12$$

$$= 12 + 3 - 12 = \boxed{3}$$

(c) The given quantity is the slope of the tangent line to the graph of $f(x)$ at the point $(1, f(1))$. Near that point, the graph is a line segment with slope 1, so

$$\lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} = \boxed{1.}$$

Alternately, you can recognize that $f(x) = x - 2$ near $x = 1$, and calculate the limit algebraically:

$$\lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1} \frac{(x-2) - (-1)}{x - 1} = \lim_{x\to 1} \frac{x - 1}{x - 1} = \lim_{x\to 1} 1 = 1.$$

(d) The graph tells us that

$$\lim_{x\to 3^+} f(x) = \boxed{-1.}$$

That is, as x approaches 3 from the right (i.e., decreases towards 3), $f(x)$ approaches -1. Note that it does not matter that $f(3) = 3$.

(e) $\lim_{x\to 3} f(x)$ $\boxed{\text{does not exist,}}$ since

$$\lim_{x\to 3^+} f(x) \neq \lim_{x\to 3^-} f(x).$$

#2. [4 pts] Use a table of values to make an intelligent guess about the value of

$$\lim_{x\to 0} (x^2)^{\sin x} .$$

Here is a typical possibility for the table (your values may differ):

x	$f(x) \approx$
0.1	0.63144
0.01	0.91201
0.001	0.98628
0.0001	0.99816
0.00001	0.99977

x	$f(x) \approx$
−0.1	1.58368
−0.01	1.09648
−0.001	1.01391
−0.0001	1.00184
−0.00001	1.00023

The data strongly suggests that the given limit equals $\boxed{1}$ (which in fact it does).

#3. [3 pts] Without using a table of values, find the exact value of $\displaystyle\lim_{x\to 1} \frac{\left(\frac{1}{x} - 1\right)}{x-1}$.

$$\lim_{x\to 1} \frac{\left(\frac{1}{x} - 1\right)}{x-1} = \lim_{x\to 1} \frac{\left(\frac{1-x}{x}\right)}{x-1} = \lim_{x\to 1} \frac{1-x}{x(x-1)} = \lim_{x\to 1} (-x) = \boxed{-1.}$$

#4. [3 pts] Use your answer to #3 to fill in the blanks...

If we define $h(x) = \frac{1}{x}$, then the given expression is precisely

$$\lim_{x\to 1} \frac{h(x) - h(1))}{x-1} ,$$

which indicates how to fill in the blanks:

> The slope of the tangent line to the graph of $\underline{y = \frac{1}{x}}$ at the point $\underline{(1, 1)}$ is $\underline{-1}$.

Bonus problem [4 pts]: Give an example of a function $f(x)$ and a value b such that $\lim_{x\to b} f(x)$ is undefined, but $\lim_{x\to b} \left(f(x)^2\right) = 1$.

To make this happen, you need to have

$$\lim_{x\to b+} f(x) = 1 \qquad \text{and} \qquad \lim_{x\to b-} f(x) = -1$$

(or vice versa), so that $\lim_{x\to b} f(x)$ does not exist, but also so that

$$\lim_{x\to b+} f(x)^2 = 1^2 = 1 \qquad \text{and} \qquad \lim_{x\to b-} f(x)^2 = (-1)^2 = 1,$$

so $\lim_{x\to b} f(x)^2 = 1$. One example is the function $f(x)$ shown in Problem #1, with $b = 3$. Another possibility is $f(x) = |x|/x$ (with domain $(-\infty, 0) \cup (0, \infty)$), $b = 0$.

Notice, however, that $f(x) = \sqrt{x}$, $b = 1$ does not work (as suggested by many of you). This is not a bad idea, butthe problem is that $f(x)^2$ is defined only where $f(x)$ is, namely on the domain $[0, \infty)$, so $\lim_{x\to -1} f(x)^2$ doesn't exist. (That is, $(\sqrt{x})^2 \neq x$ for $x < 0$.)