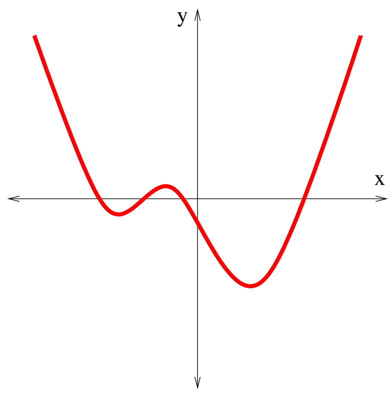
Problems #1 and #2 refer to the following graph of a function $y = f(x)$ (over part of its domain).



#1. [3 pts] Explain why $f(x)$ cannot be a polynomial of degree 2.

The graph of a polynomial of degree n has at most $n-1$ turning points. This graph has three turnarounds, so if $f(x)$ is a polynomial then its degree must be at least 4.

Alternately, the graph of a degree-2 polynomial is a parabola, which this graph clearly isn't.

#2. [3 pts] Explain why $f(x)$ cannot be a polynomial of degree 5.

The number of turning points in the graph of a polynomial of degree n has the *opposite parity* from n. This graph has three turnarounds, so if $f(x)$ is a polynomial then its degree must be even, hence can't be 5.

Alternately, the graph shown has its y-values tending to $+\infty$ on both the far left and far right. If $f(x)$ were a polynomial of odd degree, then it would tend to $-\infty$ on the left and $+\infty$ on the right, or vice versa.

#3. [4 pts] Determine the domain and range of the function $f(x) = \log_3(2 + \sin x)$.

We can evaluate $2 + \sin x$ for all real x, always obtaining a number in the interval $[1, 3]$. There is no problem taking the logarithm (with any base) of such a number, so the domain of $f(x)$ is $\mathbb{R}$. Since $\log_3$ is an increasing function and $1 \leq 2 + \sin x \leq 3$, the range of $f(x)$ is

$$[\log_3(1),\ \log_3(3)] \ = \ [0, 1].$$

#4. [4 pts] Explain how the graph of the function $a(x) = \ln(x^3)$ can be obtained from the graph of the function $b(x) = \ln x$ by an appropriate graphical transformation (e.g., shifting, stretching, reflecting, etc.) You don't have to show the actual graphs.

The rules for manipulating logarithms tell us that

$$a(x) = \ln(x^3) = 3 \ln x = 3 \cdot b(x).$$

Therefore, the graph of $a(x)$ can be obtained by vertically stretching the graph of $a(x)$ by a factor of 3.

Problems #5 and #6 refer to the functions

$$g(z) = \frac{1}{1-z} \qquad \text{and} \qquad h(z) = \frac{z}{z-1}\,.$$

#5. [3 pts] Find a formula for $g^{-1}(z)$.

Set $y = g(z) = \dfrac{1}{1-z}$ and solve for z in terms of y:

$$\begin{aligned}
y &= \frac{1}{1-z} \\
y(1-z) &= 1 \\
y - yz &= 1 \\
-yz &= 1-y \\
z &= \frac{1-y}{-y} = \frac{y-1}{y}.
\end{aligned}$$

A common mistake was to calculate $(g(z))^{-1}$ (that is, $\dfrac{1}{g(z)}$) instead. Remember that the notation $g^{-1}(z)$ means the *inverse function* of g.

#6. [3 pts] Find a formula for $(h \circ g)(z)$.

$$\begin{aligned}
(h \circ g)(z) &= h(g(z)) \\
&= h\left(\frac{1}{1-z}\right) \\
&= \frac{\frac{1}{1-z}}{\frac{1}{1-z} - 1} \\
&= \frac{\frac{1}{1-z}}{\frac{1}{1-z} - \frac{1-z}{1-z}} \\
&= \frac{\frac{1}{1-z}}{\frac{z}{1-z}} \\
&= \frac{1}{z}.
\end{aligned}$$

A common mistake was to get the order of operations wrong and calculate $(g \circ h)(z)$ instead. Remember that when composing functions, evaluate from the inside out.

#7. [4 bonus points] Again, let $g(z) = \dfrac{1}{1-z}$. Find the simplest possible formula for

$$(g \circ g)(z).$$

(In case you don't want to count them, there are a total of 24 g's.)

There must be a better way of doing this problem than using three sheets of paper to simplify this expression. Let's just calculate $(g \circ g)(z) = g(g(z))$ and see what happens:

$$\begin{aligned} g(g(z)) &= g\left(\frac{1}{1-z}\right) \\ &= \frac{1}{1 - \frac{1}{1-z}} \\ &= \frac{1}{\frac{1-z}{1-z} - \frac{1}{1-z}} \\ &= \frac{1}{\frac{-z}{1-z}} \\ &= \frac{1-z}{-z} = \frac{z-1}{z}. \end{aligned}$$

But this should look familiar from Problem 5: it's also $g^{-1}(z)$. Well, if

$$g(g(z)) = g^{-1}(z),$$

then applying g to both sides tells us that

$$g(g(g(z))) = g(g^{-1}(z)) = z.$$

That is, applying g three times to z gives z back. That means that repeatedly applying g a total of n times, where n is any multiple of 3, also gives z. Therefore,

$$(g \circ g)(z) = z.$$