Mathematics 141 Groupwork (10/28/05)

Problem #1: To get you warmed up, here's a relatively straightforward problem: evaluate

$$\lim_{x \to 0} \frac{x - \ln(x+1)}{1 - \cos 2x}.$$

Problem #2: Let X be a regular polygon with n sides, and let r be the "radius" of X, that is, the distance from the center of X to any one of its vertices. Let A be the area of X.

(a) Express A as a function of r and n.

(b) Before you do any more calculation, make an educated guess about the limit of A as $n \to \infty$.

(c) Back up your guess in part (b) by evaluting the limit exactly.

(If you're having a hard time guessing what the limit ought to equal, it is okay to calculate the limit first and then to think about why your answer makes sense.)

Problem #3: Let N be the population of the world, and let p be the probability that two randomly chosen people A and B have ever shaken hands. (Note that p is probably not a constant, but a function of N. If so, then $p = p(N)$ is almost certainly be a *decreasing* function of N.)

Is there an "antisocial" person somewhere who has never shaken hands with anyone else?

The probability that any *particular* person is antisocial can be shown to be

$$(1 - p(N))^{N-1}.$$

This expression can be rather unpleasant to evaluate, particularly since N is currently something like 6,475,087,673. Fortunately, limits come to the rescue: for such a large value of N, the probability can be be estimated very closely by taking the limit as $N \to \infty$; call this limit Y. We can actually simplify the expression by replacing the exponent $N - 1$ with N. That is,

$$Y = \lim_{N \to \infty} (1 - p(N))^N.$$

(a) Evaluate Y if $p(N) = c/N$, where c is a positive constant. (Hint: This is actually a special case of one of the homework problems from §4.5.)

(b) Evaluate Y if $p(N) = c/N^2$.

(c) Evaluate Y if $p(N) = \dfrac{\ln N}{N}$.

Note: This is an example of the very general and powerful idea of a *random graph*, which can be used to study many networks arising in nature. Other examples include hydrogen bonding between water molecules in a block of ice; the spread of Dutch elm disease between trees in a forest; the most efficient way to locate wireless Internet routers; and many others.

Problem #4: Suppose that $f(x)$ and $g(x)$ are functions such that

$$\lim_{x\to a} f(x) = +\infty, \qquad \lim_{x\to a} g(x) = +\infty,$$

and

$$\lim_{x\to a} [f(x) - g(x)] = r,$$

where r is some real number.

(a) What can you say about the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)}?$$

Does it always exist? If so, what do you think its value is? (You might want to cite some examples.)

(b) Can you prove that your guess in (a) is correct? (Hint: Dirty tricks may be required.)

Problem #5 (Harder!) Another limit that arises in the theory of random graphs is

$$\lim_{n\to\infty} n\left(1 - \frac{c\ln n}{n}\right)^n,$$

where c is a positive constant.

Show that

$$\lim_{n\to\infty} n\left(1 - \frac{c\ln n}{n}\right)^n = \begin{cases} \infty & \text{if } 0 < c < 1, \\ 1 & \text{if } c = 1, \\ 0 & \text{if } c > 1. \end{cases}$$