I'll use the following (nonstandard) notation throughout: whenever I apply L'Hôpital's Rule, I'll use the symbol $\stackrel{\text{LH}}{=}$ to alert you that that's what I'm doing!

Problem #1: Evaluate

$$\lim_{x\to 0} \frac{x - \ln(x+1)}{1 - \cos 2x}.$$

This is a indefinite form of type 0/0, so we can apply L'Hôpital's Rule to obtain

$$\lim_{x\to 0} \frac{1 - \frac{1}{x+1}}{2\sin 2x}.$$

Cleaning this up gives

$$\lim_{x\to 0} \frac{x}{2(x+1)\sin(2x)}.$$

This is still a 0/0 form, so we apply L'Hôpital's Rule again (first pulling out the constant 1/2), obtaining

$$\frac{1}{2}\lim_{x\to 0} \frac{1}{\sin(2x) + 2(x+1)\cos(2x)}.$$

Now *this* limit can be evaluated by plugging in $x = 0$. We get $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ as the final answer.

Problem #2: Let X be a regular polygon with n sides, and let r be the "radius" of X, that is, the distance from the center of X to any one of its vertices. Let A be the area of X.

(a) Express A as a function of r and n.

Slice the polygon up into n isosceles triangles as shown. Each triangle has base $2r\sin\theta$ and height $r\cos\theta$, hence area $r^2 \sin\theta\cos\theta$. Since there are n triangles and $\theta = \pi/n$, we obtain

$$A = nr^2\ \sin\frac{\pi}{n}\ \cos\frac{\pi}{n}.$$

(b) Before you do any more calculation, make an educated guess about the limit of A as $n \to \infty$.

The larger n gets, the more the polygon looks like a circle with radius r. So a logical guess would be

$$\lim_{n\to\infty} nr^2 \sin\frac{\pi}{n}\cos\frac{\pi}{n} = \pi r^2.$$

(c) Back up your guess in part (b) by evaluting the limit exactly.

$$
\begin{aligned}
\lim_{n\to\infty} nr^2 \sin(\pi/n)\cos(\pi/n) &= \left(r^2 \lim_{n\to\infty} \cos(\pi/n)\right)\left(\lim_{n\to\infty} n\sin(\pi/n)\right) \\
&= r^2 \lim_{n\to\infty} \frac{\sin(\pi/n)}{n^{-1}} \\
&\overset{\text{LH}}{=} r^2 \lim_{n\to\infty} \frac{(\cos(\pi/n))\,(-\pi n^{-2})}{-n^{-2}} \\
&= r^2 \lim_{n\to\infty} \pi\cos(\pi/n) \\
&= \pi r^2,
\end{aligned}
$$

as expected.

Problem #3: Let N be the population of the world, and let p be the probability that two randomly chosen people A and B have ever shaken hands. (Note that p is probably not a constant, but a function of N. If so, then $p = p(N)$ is almost certainly be a *decreasing* function of N.)

Is there an "antisocial" person somewhere who has never shaken hands with anyone else?

The probability that any *particular* person is antisocial can be shown to be

$$(1 - p(N))^{N-1}.$$

This expression can be rather unpleasant to evaluate, particularly since N is currently something like 6,475,087,673. Fortunately, limits come to the rescue: for such a large value of N, the probability can be be estimated very closely by taking the limit as $N \to \infty$; call this limit Y. We can actually simplify the expression by replacing the exponent $N-1$ with N. That is,

$$Y = \lim_{N\to\infty} (1 - p(N))^N.$$

(a) Evaluate Y if $p(N) = c/N$, where c is a positive constant. (Hint: This is actually a special case of one of the homework problems from §4.5.)

The limit Y has the indefinite form 1^∞. To evaluate Y, we replace the expression $(1 - p(N))^N$ with its logarithm, in order to convert it to one of the indefinite forms governed by L'Hôpital's Rule:

$$
\begin{aligned}
Z = \lim_{N\to\infty} \ln(1 - c/N)^N &= \lim_{N\to\infty} N\ln(1 - c/N) \\
&= \lim_{N\to\infty} \frac{\ln(1 - cN^{-1})}{N^{-1}} \\
&\overset{\text{LH}}{=} \lim_{N\to\infty} \frac{\frac{1}{1-cN^{-1}}(cN^{-2})}{-N^{-2}} \\
&= \lim_{N\to\infty} \frac{-c}{1 - cN^{-1}} \\
&= -c,
\end{aligned}
$$

since $cN^{-1} \to 0$ as $N \to \infty$. Therefore $Y = e^Z = e^{-c}$.

(b) Evaluate Y if $p(N) = c/N^2$.

The same technique as before yields

$$\begin{aligned} Z &= \lim_{N\to\infty} \ln(1 - c/N^2)^N = \lim_{N\to\infty} N \ln(1 - c/N^2) \\ &= \lim_{N\to\infty} \frac{\ln(1 - cN^{-2})}{N^{-1}} \\ &\overset{\text{LH}}{=} \lim_{N\to\infty} \frac{\frac{1}{1-cN^{-2}}(2cN^{-3})}{-N^{-2}} \\ &= \lim_{N\to\infty} \frac{-2cN^{-1}}{1 - cN^{-2}} \\ &= 0. \end{aligned}$$

Therefore $Y = e^Z = e^0 = 1$.

(c) Evaluate Y if $p(N) = \dfrac{\ln N}{N}$.

Yet again, we evaluate

$$\begin{aligned} Z &= \lim_{N\to\infty} \ln\left(1 - \frac{\ln N}{N}\right)^N = \lim_{N\to\infty} N \ln\left(1 - \frac{\ln N}{N}\right) \\ &= \lim_{N\to\infty} \frac{\ln\left(1 - \frac{\ln N}{N}\right)}{N^{-1}} \\ &\overset{\text{LH}}{=} \lim_{N\to\infty} \frac{\left(\frac{1}{1-\frac{\ln N}{N}}\right)\left(-\frac{1-\ln N}{N^2}\right)}{-N^{-2}} \\ &= \lim_{N\to\infty} \frac{1 - \ln N}{1 - \frac{\ln N}{N}}. \end{aligned}$$

As $N \to \infty$, $\ln N \to \infty$, so the numerator of this expression tends to $-\infty$. However, $\lim_{N\to\infty} \frac{\ln N}{N} = 0$, so the denominator of this expression tends to 1. It follows that $Z = -\infty$. We'd like to say that $Y = e^Z$, but it's not really correct to write $Y = e^{-\infty}$. To be precise, the expression

$$\ln\left(1 - \frac{\ln N}{N}\right)^N$$

decreases without bound as $N \to \infty$, which means that

$$\left(1 - \frac{\ln N}{N}\right)^N$$

must tend to zero. Therefore $Y = 0$.

Note: This is an example of the very general and powerful idea of a *random graph*, which can be used to study many networks arising in nature. Other examples include hydrogen bonding between water molecules in a block of ice; the spread of Dutch elm disease between trees in a forest; the most efficient way to locate wireless Internet routers; and many others.

Problem #4: Suppose that $f(x)$ and $g(x)$ are functions such that

$$\lim_{x \to a} f(x) = +\infty, \qquad \lim_{x \to a} g(x) = +\infty,$$

and

$$\lim_{x \to a} [f(x) - g(x)] = r,$$

where r is some real number.

(a) What can you say about the limit

$$\lim_{x \to a} \frac{f(x)}{g(x)}?$$

Does it always exist? If so, what do you think its value is? (You might want to cite some examples.)

The easiest way to construct an example is to let $f(x) = g(x) + r$, in which case it is not hard to see that the limit in question must be 1. Intuitively, this makes sense—if the difference between $f(x)$ and $g(x)$ is bounded by a constant, they should be growing at the same rate.

(b) Can you prove that your guess in (a) is correct? (Hint: Dirty tricks may be required.)

Try this:

$$\begin{aligned}
\lim_{x \to a} \frac{f(x)}{g(x)} &= \lim_{x \to a} \frac{f(x) - g(x) + g(x)}{g(x)} \\
&= \lim_{x \to a} \frac{f(x) - g(x)}{g(x)} + \lim_{x \to a} g(x)g(x) \\
&= \lim_{x \to a} \frac{r}{g(x)} + 1 \\
&= 0 + 1 \ = \ 1.
\end{aligned}$$

Problem #5 (Harder!) Another limit that arises in the theory of random graphs is

$$\lim_{n \to \infty} n \left(1 - \frac{c \ln n}{n}\right)^n,$$

where c is a positive constant.

Show that

$$\lim_{n \to \infty} n \left(1 - \frac{c \ln n}{n}\right)^n = \begin{cases} \infty & \text{if } 0 < c < 1, \\ 1 & \text{if } c = 1, \\ 0 & \text{if } c > 1. \end{cases}$$

As is so often the case, we will calculate the limit of the logarithm of the expression we are interested in; that is,

$$\lim_{n\to\infty} \ln\left[n\left(1-\frac{c\ln n}{n}\right)^n\right].$$

First, we do some algebraic manipulation.

$$\begin{aligned}
\lim_{n\to\infty} \ln\left[n\left(1-\frac{c\ln n}{n}\right)^n\right] &= \lim_{n\to\infty} \ln\left[n\left(\frac{n-c\ln n}{n}\right)^n\right] \\
&= \lim_{n\to\infty} \ln\left[n\left(\frac{(n-c\ln n)^n}{n^n}\right)\right] \\
&= \lim_{n\to\infty} \ln\left[\frac{(n-c\ln n)^n}{n^{n-1}}\right] \\
&= \lim_{n\to\infty} \left[n\ln(n-c\ln n)-(n-1)\ln n\right].
\end{aligned}$$

Now, we set $m = \ln n$, so that $n = e^m$. As $n \to \infty$, so does m, so we can rewrite this expression in terms of m rather than n, and then do some more algebra:

$$\begin{aligned}
&= \lim_{m\to\infty} \left[e^m\ln(e^m-cm)-(e^m-1)m\right] \\
&= \lim_{m\to\infty} \left[e^m(\ln(e^m-cm)-m)+m\right] \\
&= \lim_{m\to\infty} \frac{\ln(e^m-cm)-m+me^{-m}}{e^{-m}}. \qquad (1)
\end{aligned}$$

I claim that (1) is an indeterminate form of type 0/0. The denominator certainly tends to zero as $m \to \infty$, but the numerator is itself an indeterminate form (technically of type $\infty - \infty$). However,

$$\lim_{m\to\infty} me^{-m} = \lim_{m\to\infty} \frac{m}{e^m} \stackrel{\text{LH}}{=} \lim_{m\to\infty} \frac{1}{e^m} = 0, \qquad (2)$$

and

$$\begin{aligned}
\lim_{m\to\infty} \ln(e^m-cm)-m &= \ln \lim_{m\to\infty} \exp\left(\ln(e^m-cm)-m\right) \\
&= \ln \lim_{m\to\infty} \frac{e^m-cm}{e^m} \\
&\stackrel{\text{LH}}{=} \ln \lim_{m\to\infty} \frac{e^m-c}{e^m} \\
&\stackrel{\text{LH}}{=} \ln \lim_{m\to\infty} \frac{e^m}{e^m} = \ln 1 = 0,
\end{aligned}$$

proving my claim. The upshot is that we can apply L'Hôpital's Rule to (1):

$$\lim_{m\to\infty} \frac{\ln(e^m-cm)-m+me^{-m}}{e^{-m}} \stackrel{\text{LH}}{=} \lim_{m\to\infty} \frac{\frac{e^m-c}{e^m-cm}-1+e^{-m}-me^{-m}}{-e^{-m}}.$$

Now we do some more algebra:

$$\begin{aligned}\lim_{m\to\infty}\frac{\frac{e^m-c}{e^m-cm}-1+e^{-m}-me^{-m}}{-e^{-m}}. &= \lim_{m\to\infty}\frac{e^m-c+(e^m-cm)(-1+e^{-m}-me^{-m})}{e^{-m}(cm-e^m)}\\ &= \lim_{m\to\infty}\frac{e^m-c+(-e^m+1-m+cm-cme^{-m}+cm^2e^{-m})}{(e^m-cm)(-e^{-m})}\\ &= \lim_{m\to\infty}\frac{-c+1-m+cm-cme^{-m}+cm^2e^{-m}}{cme^{-m}-1}\end{aligned}$$

This is **not** an indeterminate form—its denominator tends to -1 as $m\to\infty$, so it equals

$$-\lim_{m\to\infty} -c+1-m+cm-cme^{-m}+cm^2e^{-m}.$$

In this expression, the two terms that involve e^{-m} tend to zero as $m\to infty$ (this is essentially the same calculation is (2)), so we can throw them away. Distributing the minus sign, we wind up with

$$\lim_{m\to\infty}(1-c)m+c-1,$$

which is clearly equal to

$$\begin{cases}-\infty & \text{if } c>1,\\ 0 & \text{if } c=1,\\ +\infty & \text{if } c<1.\end{cases} \tag{3}$$

Remember, this is the limit of the **logarithm** of the actual expression we are interested in. The desired conclusion is now immediate from (3). (Phew!)