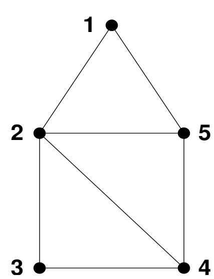
My sources:

- Norman Biggs, "Chip-firing and the critical group of a graph", J. Alg. Combin. 9 (1999), 25–45
- Discussions with Gregg Musiker

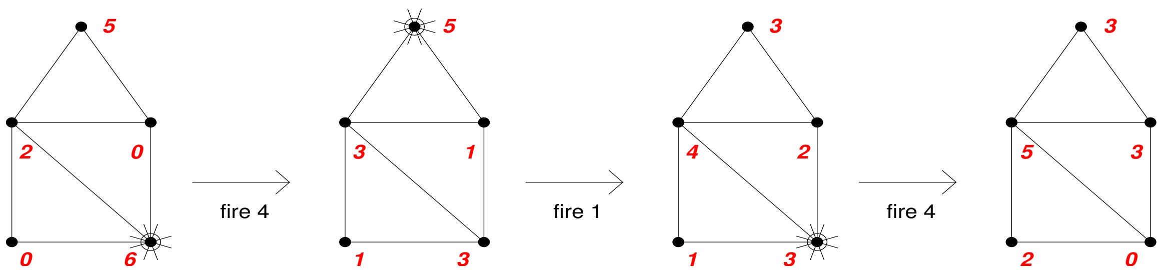
The Chip-Firing Game

- Start with a finite, simple, connected graph $G = (V, E)$, with $V = [n] = \{1, 2, \ldots, n\}$ and $E = \{e_1, \ldots, e_m\}$. Let d_i be the *degree* of vertex i (the number of adjacent vertices).
- Place c_i *chips* on each vertex i. Record the number of chips by a vector $c = (c_1, \ldots, c_n) \in \mathbb{N}^n$, called a *configuration*.
- Vertices are altruistic (they want to donate chips to their neighbors). They're also egalitarian (they like all their neighbors equally).
- Vertex i is *ready* if $c_i \geq d_i$. Such a vertex can *fire* by distributing one chip to each of its neighbors.

For example, let G be as follows:



Here is a possible sequence of firings:



One more thing: there is one special vertex q called the *bank*. Unlike its pals, the bank is a miser: it doesn't fire, but just sits there collecting chips. Eventually, so many chips accumulate at the bank that no other vertex can fire. Such a configuration, is called *stable*.

At this point, the bank begrudgingly fires, in order to get the economy going again. It can even go into debt: that is, c_q is allowed to be negative, although no other c_i is. An important point is that the bank fires *if and only if* the current configuration is stable.

What does this have to do with anything?

The *Laplacian matrix* of G is the $n \times n$ symmetric matrix L with entries

$$L_{ij} = \begin{cases} d_i & \text{if } i = j, \\ -1 & \text{if } ij \in E, \\ 0 & \text{otherwise.} \end{cases}$$

For the graph above we have

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}.$$

Firing a vertex is equivalent to subtracting the corresponding row of the Laplacian from the configuration. To put it another way, if one configuration can be obtained from another by firing a sequence of vertices, then they represent the same element of the cokernel of the Laplacian (i.e., the quotient of $\mathbb{Z}^n$ by the rowspace of L).

Proposition 1. (1) *The all-ones vector $\vec{1}$ is a nullvector of L.*
(2) *Provided that G is connected, the rank of L is $n - 1$.*

Proof. Denote the i^{th} row of L by L_i.

(1) The assertion is equivalent to saying that each L_i sums to 0. By definition, L_{ii} equals the number of -1 entries in L_i.
(2) Suppose that $\sum_{i=1}^{n} c_i L_i = 0$. Let $k = \min\{c_i\}$ and $b_i = c_i + k$. Then $\min\{b_i\} = 0$, and

$$\sum_{i=1}^{n} b_i X_i \; = \; \left(\sum_{i=1}^{n} c_i X_i \right) + L\vec{1} \; = \; \vec{0}.$$

Let $X = \{i \mid b_i > 0\}$. Suppose that $X \neq \emptyset$. Since G is connected, there is some vertex $k \notin X$ with a neighbor $j \in X$. But then the k^{th} entry of $\sum_{i \in X} c_i L_i$ is negative (there is a negative contribution from $i = j$ and a nonpositive contribution from each other i), a contradiction. Therefore $X = \emptyset$, so $b_i = 0$ for all i, so all the c_i's are equal. So $\vec{1}$ spans the nullspace of L. $\square$

To get all homological about it, we have a chain complex

$$\mathbb{Z}^n \xrightarrow{L} \mathbb{Z}^n \xrightarrow{S} \mathbb{Z} \to 0$$

where $S(c) = c \cdot \vec{1} = c_1 + \cdots + c_n$. Assertion (1) of the Proposition says that $\operatorname{Ker}(S) \supseteq \operatorname{Im}(L)$, and assertion (2) says that the abelian group $\operatorname{Ker}(S) \operatorname{Im}(L)$ is finite.

Fact: $|\operatorname{Ker}(S)/\operatorname{Im}(L)|$ = number of spanning trees of G.

Goal: Find a bijection between $\operatorname{Ker}(S)/\operatorname{Im}(L)$ and certain canonical ("critical") chip configurations.

The bank vertex q is allowed to go into deficit, so we don't really care about the number of chips on it. Therefore, we may as well work with configurations $c \in \operatorname{Ker}(S)$, i.e., such that $\sum_{i=1}^n c_i = 0$.

Definition/Notation: Let c be a configuration and $X = (x_1, \ldots, x_r)$ a sequence of vertices.

- c is **q-nonnegative** if $\sum_{i=1}^n c_i = 0$ and $c_i \geq 0$ for all $i \neq n$.
- $c(X)$ or $c(x_1, \ldots, x_r)$ denotes the configuration obtained after the vertices $x_1, \ldots, x_r$ fire. In terms of the Laplacian,

$$c(X) = c - \sum_{i=1}^{n} L_{x_i}. \tag{1}$$

- X is **proper** (relative to c) if it does not contain q and no non-bank vertex goes into debt along the way. It is **q-proper** if it is proper and does not contain q.
- A q-nonnegative configuration c is *stable* if no non-bank vertex can fire, i.e., $c_i < d_i$ for all $i \neq q$. It is *recurrent* if there is a nontrivial firing sequence X such that $c(X) = c$, and it is *critical* if it is both stable and recurrent.

Our goal is the following result:

Theorem 2. *Every equivalence class in* $\operatorname{Ker}(S)/\operatorname{Im}(L)$ *contains exactly one critical configuration.*

Sketch of proof:

(1) Existence (i.e., proving that every equivalence class contains at least one critical configuration) is fairly easy.
(2) For uniqueness, we want to show that all proper firing sequences ultimately lead to the same critical configuration. First, we show that if you start with an unstable configuration, then it doesn't matter in which order you fire the vertices; you always get to the same stable configuration.
(3) This means that we can model the chip-firing game by a directed graph Γ whose vertices are stable configurations, whose edges record which configurations lead to which other ones, and where each vertex has out-degree exactly 1. Note that a stable configuration is critical iff it is part of a directed cycle in Γ.
(4) Next we rule out all cycles of length > 1. That means that Γ looks like a bunch of rooted trees, with all edges directed towards the root, plus loops on the roots.
(5) *Finally, we need to show that each component of* Γ *corresponds to a different congruence class modulo the Laplacian.*

Lemma 3. *Any sufficiently long q-proper firing sequence produces a stable configuration.*

Proof. Let $c = (c_1, \dots, c_n)$ be q-stable, let X be a q-proper firing sequence, and let $b = (b_1, \dots, b_n) = c(X)$. There are only finitely many possibilities for b, because $b_i \geq 0$ for $i \neq q$ and $\sum_{i \neq q} b_i \leq \sum_{i \neq q} c_i$. Therefore, it suffices to prove that no configuration occurs more than once during the firing sequence.

Let Y be the set of vertices that fire, and let $Z = V \setminus Y$. Then Y and Z are both nonempty (because $q \in Z$). No chip ever moves from Z to Y, but since G is connected, there is at least one edge e between Y and Z, so at least one chip moves from Z to Y. Therefore, $\sum_{i \in Y} c_i < \sum_{i \in Y} b_i$, and consequently $c \neq b$. ☐

Corollary 4. *Every equivalence class in* $\operatorname{Ker}(S)/\operatorname{Im}(L)$ *contains at least one stable configuration.*

Proof. Given any configuration, fire n repeatedly until you obtain a q-nonnegative configuration. Then apply the Lemma. ☐

Lemma 5. *Every equivalence class in* $\operatorname{Ker}(S)/\operatorname{Im}(L)$ *contains at least one critical configuration.*

Proof. The number of stable configurations is finite (because $0 \leq c_i < d_i$ for all $i \neq n$, and c_n is determined by $c_1, \ldots, c_n$). So start with a stable configuration, fire the bank until the configuration is unstable (i.e., some other vertex is ready), reduce down to a stable configuration, and repeat. Some stable configuration will eventually recur, hence it is critical. $\square$

Definition/Notation: Let X, Y be firing sequences.

- The concatenation of X and Y is denoted (X, Y).
- The *score vector* of X is $\alpha = (\alpha_1, \ldots, \alpha_n)$, where vertex i occurs α_i times in X.
- If $\beta \in \mathbb{N}^n$ (e.g., the score vector of Y) respectively, then $X \setminus \beta$ denotes the sequence obtained from X by deleting the first β_i occurrences of each i (or all occurrences if $\beta_i > \alpha_i$).

Proposition 6. *c: q-nonnegative configuration*
X, Y: q-proper with respect to c
α, β: their score vectors

Then there exists a q-nonnegative configuration — namely, $c(Y, X \setminus \beta)$ — reachable from both $c(X)$ and $c(Y)$.

Proof. If c is stable then this is trivial, since X, Y must be empty.

Otherwise, let $Z = (Y, X \setminus \beta)$.

Score vector γ of Z is

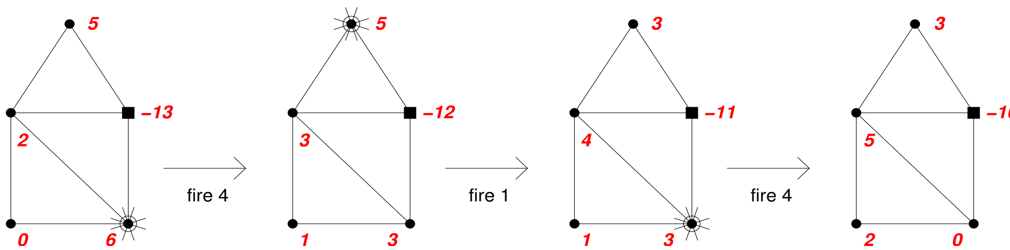
$$\gamma_i = \max\{\alpha_i, \beta_i\} \qquad \text{for } i \neq q$$

$=$ score vector of $Z' = (X, Y \setminus \alpha)$

So $c(Z) = c(Z')$ is reachable from both $c(X)$ and $c(Y)$.

We need to prove that Z and Z' are q-proper.
Suffices to prove it for Z.

Example 1. Let G be as above, and let $q = 5$ (the square vertex). Ignore c_5 (remember, we are always working with configurations in $\operatorname{Ker}(S)$).



So $c = (5, 2, 0, 6)$, and the firing sequence is $X = (4, 1, 4)$. Thus

$$\begin{aligned} c(4) &= (5, 3, 1, 3), \\ c(4, 1) &= (3, 4, 1, 3), \\ c(X) &= (3, 5, 2, 0). \end{aligned}$$

Thus X is q-nonnegative. Here's another firing sequence: $Y = (1, 1, 2, 4, 1)$. This is q-nonnegative because

$$\begin{aligned} c(1) &= (3, 3, 0, 6), \\ c(1, 1) &= (1, 4, 0, 6), \\ c(1, 1, 2) &= (2, 0, 1, 7) \\ c(1, 1, 2, 4) &= (2, 1, 2, 4), \\ c(Y) &= (0, 2, 2, 4). \end{aligned}$$

So

$$\begin{aligned} X \setminus \beta &= (4), & (Y, X \setminus Y) &= (1, 1, 2, 4, 1, 4), \\ Y \setminus \alpha &= (1, 2, 1), & (X, Y \setminus X) &= (4, 1, 4, 1, 2, 1). \end{aligned}$$

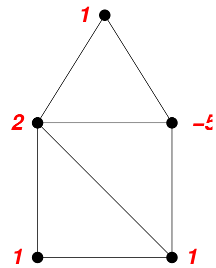
Then $c(Y, X \setminus Y) = c(Y)(4) = (0, 3, 3, 1)$. Meanwhile,

$$\begin{aligned} c(X, 1) &= (1, 6, 2, 0), \\ c(X, 1, 2) &= (2, 2, 3, 1), \\ c(X, 1, 2, 1) &= (0, 3, 3, 1). \end{aligned}$$

So, indeed, $(X, Y \setminus \alpha)$ and $(Y, X \setminus \beta)$ are q-proper, and of course $c(X, Y \setminus \alpha) = c(Y, X \setminus \beta)$. This sequence is not stable, because, e.g., vertex 3 can fire.

By the way, if we keep firing, we'll eventually get a stable configuration, as predicted by Lemma 3.

fire 3 to get $(0, 4, 1, 2)$;
fire 2 to get $(1, 0, 2, 3)$;
fire 4 to get $(1, 1, 3, 0)$;
fire 3 to get $(1, 2, 1, 1)$ (which is stable).



Back to the proof of the general case. Essentially, we induct on the length of $X \setminus \beta$, although it is easier to state the proof algorithmically.

If it's zero, then $Z = Y$ is q-proper and we're done. Otherwise:

Let i be the first entry in $X \setminus \beta$.

- (Here $X \setminus \beta = (4)$, so $i = 4$.)

In particular, $\beta_i < \alpha_i$.

- (Here, $\beta_4 = 1$ and $\alpha_4 = 2$.)

Let $X' =$ initial subsequence of X that ends just before the $(\beta_i + 1)^{st}$ appearance of i in X. So X' is certainly q-proper.

- (Here, we want X' to end just before the second 4, so $X' = (4, 1)$.)

Let α' be the score of X'. By construction, X' contains exactly β_i copies of i, so

$$\alpha'_i = \beta_i.$$

Also, since i is the first entry in $X \setminus \beta$, we must have

$$\alpha'_j \leq \beta_j \quad \forall j \neq i.$$

I claim that

$$c(Y)_i \geq c(X')_i \geq d_i.$$

The first inequality follows because i itself has fired the same number of times in Y as it has in X', but its neighbors have all fired at least as many times in Y as in X, so i has been donated at least as many chips. For the second inequality, i must be ready to fire in $c(X')$ because $X = (X', i, \dots)$ is a q-proper sequence.

Therefore, (Y, i) is q-proper. We can now continue by replacing Y with (Y, i) and repeating the argument. we can show that every initial subsequence of $(Y, X \setminus \beta)$ is q-proper. $\square$

Corollary 7. *Let c be a q-nonnegative configuration, and let X and Y be q-proper firing sequences with respect to c, such that $c(X)$ and $c(Y)$ are stable. Then in fact $c(X) = c(Y)$. Therefore, there is a unique stable configuration reachable from c by a q-proper firing sequence.*

Proof. The only q-proper firing sequence following a stable sequence is the empty sequence. So $X \setminus \beta = Y \setminus \alpha$ is empty, and X and Y must have the same scores. $\square$

Define a directed graph Γ whose vertices are the stable configurations (recall that there are only finitely many of these), with edges $\overrightarrow{cb}$ whenever b can be reached from c in the chip-firing game. Thus a stable configuration is critical if and only if it is part of a (directed) cycle in Γ.

If c is stable, then the rules of the chip-firing game force us to fire q repeatedly until we reach an unstable configuration, at which point Corollary 7 takes over. Therefore, every vertex of Γ has out-degree exactly 1 (i.e., for each c, there is exactly one edge of the form $\overrightarrow{cb}$.

Proposition 8. *Γ has no cycles of length > 1.*

Proof. A cycle in Γ corresponds to a stable configuration c and a firing sequence $X = (x_1, \ldots, x_r)$ such that $c(X) = c$.

α = score vector of X: nullvector of the Laplacian, hence a multiple of $\vec{1}$. That is, each vertex (including q) appears equally many times in X.

Claim: Each $i \neq q$ appears exactly once in X between every two consecutive occurrences of q.

Suppose not. Then X has a subsequence

$$X' = (x_s = q, \ldots, i, \ldots, i)$$

where there are no occurrences of q in the "…". By choosing the shortest such subsequence, we can assume that no other vertex appears more than once in X'.

Let $b = c(x_1, \ldots, x_{s-1})$. The next vertex fired is q, so b is stable. In particular, $b_i < d_i$. But then

$$b(X')_i \;=\; b_i - 2d_i + \#\{\text{neighbors of } i \text{ in } X'\} \;<\; d_i - 2d_i + d_i = 0$$

which is illegal! This contradiction proves the claim.

That means that X must have the form

$$(\sigma_1,\ \sigma_2,\ \ldots,\ \sigma_t)$$

where each σ_j is a permutation of V, starting with q. But $c = c(\sigma_1) = c(\sigma_1, \sigma_2) = \cdots = c(X)$. So this ostensible "cycle" in X is actually just a repeated loop. □

We have now proven that every starting configuration c leads to a well-defined, unique critical configuration, which I'll call $R(c)$ (R for "reduction"). Also, if $R(c) = b$ then c and b are congruent modulo L. Therefore, for each $b \in K(G)$, the corresponding fiber of R, namely

$$R^{-1}(b) = \{c \in \operatorname{Ker}(S) \mid R(c) = b\},$$

is contained in some coset of L.

Theorem 9. *Let $K(G)$ be the set of critical configurations. The fibers of the function $R : \operatorname{Ker}(S) \to K(G)$ are exactly the cosets of L. That is, R defines a bijection*

$$\operatorname{Ker}(S)/\operatorname{Im}(L) \to K(G).$$

Proof. Suppose that two configurations c_1, c_2 are in the same coset of the Laplacian. We must show that $R(c_1) = R(c_2)$. Suppose that $c_1 - c_2 = Lf$, where $f \in \mathbb{Z}^n$. Write $f = f_1 - f_2$, where $f_1, f_2 \in \mathbb{N}^n$. Let $c_0 = c_1 - Lf_1 = c_2 - Lf_2$,

- For example, with G as above, let

$$\begin{aligned} c_1 &= (4, 1, 6, 1, -12), & c_1 - c_2 &= (3, -5, 4, -3, 1) \\ c_2 &= (1, 6, 2, 4, -13) & &= L \cdot (1, -1, 1, -1, 0), \end{aligned}$$

(you could get from c_1 to c_2 by firing vertices 1 and 3 and "unfiring" 2 and 4, if that were legal, which it isn't). Continuing the example:

$$\begin{aligned} f &= (1, -1, 1, -1, 0), & c_0 &= (2, 3, 4, 2, -11). \\ f_1 &= (1, 0, 1, 0, 0), \\ f_2 &= (0, 1, 0, 1, 0), \end{aligned}$$

Let $R(c_1) = b \in K(G)$, and let X be a legal sequence such that $c_1(X) = b$.

- In the example, $X = (1, 1, 3, 3, 3, 2, 4, 3, 2, 1, 4, 3)$, and $b = (0, 3, 0, 2)$.

We can assume that X contains each vertex i at least $f_1(i)$ times, because if necessary, we can fire the bank and reduce back down to b (remember, b is recurrent).

- In the example, we don't have to do this.

Now the proof of Proposition 6 implies that $X' := X \setminus \alpha$ is proper for c_0. Therefore
$$R(c_0) = c_0(X') = b = c_1(X) = R(c_1).$$
But the same argument implies that $R(c_2) = R(c_0)$. Therefore $R(c_1) = R(c_2)$ and we're done. $\square$

That finishes the proof of Theorem 2.

Corollary 10. *The critical configurations $K(G)$ form an group, the* critical group*, under the operation of addition followed by reduction (in the chip-firing game) to a critical configuration. This group is a finite abelian group isomorphic to* $\mathrm{Ker}(S)/\,\mathrm{Im}(L)$*. In particular, the number of critical configurations equals the number of spanning trees of G.*