Totally symmetric self-complementary plane partition matrices and polytopes

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Alternating sign matrices

Definition

Alternating sign matrices (ASM) are square matrices with the following properties:

- entries $\in \{0, 1, -1\}$
- each row and each column sums to 1
- nonzero entries alternate in sign along a row/column

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Alternating sign matrix enumeration

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Theorem (Zeilberger 1996; Kuperberg 1996)

$n \times n$ alternating sign matrices are counted by: $\displaystyle\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$.

1, 2, 7, 42, 429, 7436, 218348, 10850216, ...

This was conjectured by Mills, Robbins, and Rumsey (1983) and proved in different ways by Zeilberger (1996), Kuperberg (1996), Fischer (2006), and Fischer-Konvalinka (2020/2022).

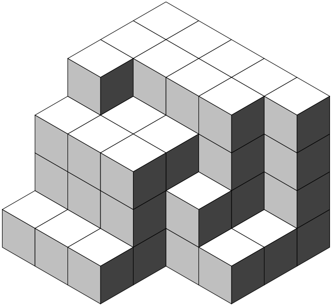
Plane partitions

Definition

A **plane partition** π is a set of positive integer lattice points (i, j, k) such that if $(i, j, k) \in \pi$, $i' \leq i$, $j' \leq j$, $k' \leq k$ then $(i', j', k') \in \pi$.

Theorem (MacMahon 1896)

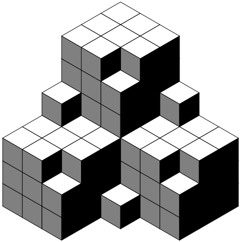
The number of plane partitions inside $a \times b \times c$ is $\displaystyle\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}$



Symmetry classes of plane partitions

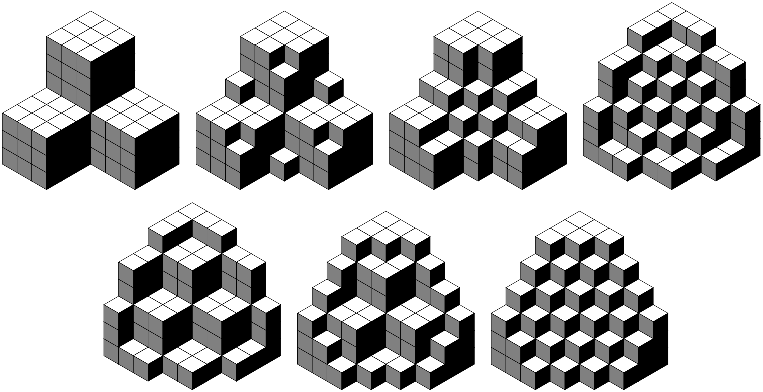
Definition

- π is **symmetric** if $(i, j, k) \in \pi$ implies $(j, i, k) \in \pi$.
- π is **cyclically symmetric** if $(i, j, k) \in \pi$ implies $(j, k, i) \in \pi$.
- π is **totally symmetric** if both of the above hold.
- π is **self-complementary** if it is equal to its complement.



Totally symmetric self-complementary plane partition (TSSCPP)

TSSCPP inside a $6 \times 6 \times 6$ box

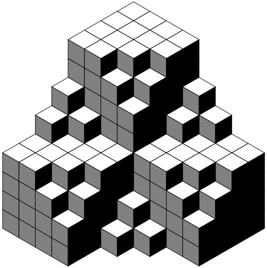


Theorem (G. Andrews 1994)

Totally symmetric self-complementary plane partitions inside a $2n \times 2n \times 2n$ box are counted by: $\displaystyle\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$.

A missing bijection

Open problem: Find a beautiful bijection between $n \times n$ ASM and TSSCPP in a $2n \times 2n \times 2n$ box.



$$?$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

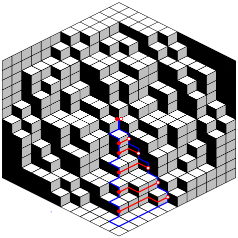
Sub-bijections known in the following cases:

- Permutation [S. 2018]
- Two-diagonal [Biane and Cheballah 2012, Bettinelli 2019]
- Gapless monotone triangle [ACGB 2011]
- 1432-key-avoiding [Huang and S. 2024]

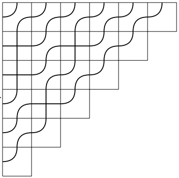
A 1432-avoiding ASM-TSSCPP bijection

Theorem (Huang-S. 2024)

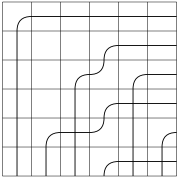
Let $\pi \in S_n$. There is an explicit weight-preserving injection φ from $TSSCPP^{red}(\pi)$ to $ASM^{red}(\pi)$. If π avoids 1432, then φ is a bijection.



$\Leftrightarrow$



$\Leftrightarrow$



$$\Leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Known alternating sign matrix bijections

ASM

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

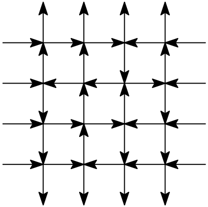
Monotone triangle

$$\begin{array}{ccccccc} & & & 3 & & & \\ & & 1 & & 4 & & \\ & 1 & & 3 & & 4 & \\ 1 & & 2 & & 3 & & 4 \end{array}$$

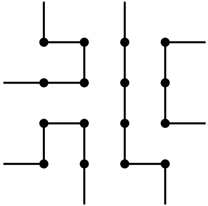
Height function

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 2 & 3 \\ 2 & 1 & 2 & 3 & 2 \\ 3 & 2 & 3 & 2 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

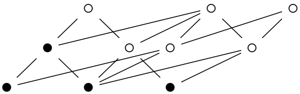
Six-vertex model



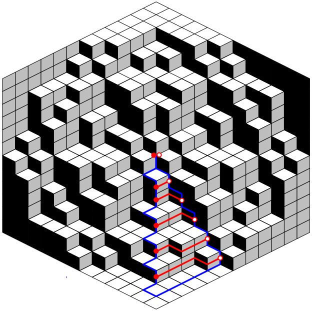
Fully-packed loop

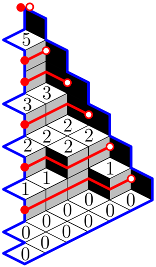


Order ideal



Known TSSCPP bijections

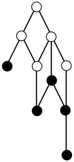




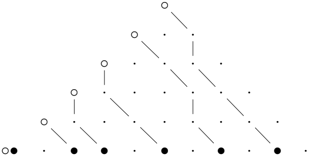
Magog triangle

6
4 5
3 4 5
2 3 5 6
1 2 3 4 6
1 2 3 4 5 6

Order ideal



Non-intersecting lattice paths



New TSSCPP object: Magog matrices

Magog triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 6 & & & & & \\
 & & & & 4 & & 5 & & & & \\
 & & & 3 & & 4 & & 5 & & & \\
 & & 2 & & 3 & & 5 & & 6 & & \\
 & 1 & & 2 & & 3 & & 4 & & 6 & \\
1 & & 2 & & 3 & & 4 & & 5 & & 6
\end{array}
\longleftrightarrow
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
$$

Magog matrix

$$
\longleftrightarrow
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 \\
1 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

Characterizing magog matrices (Holmlund and S. 2024)

ASM vs magog matrices

Properties of both:

- entries $\in \{0, 1, -1\}$
- rows and columns sum to one
- row and column partial sums ≥ 0
- column partial sums ≤ 1

Property of ASMs:

- row partial sums ≤ 1

Property of Magog matrices:

- For all $0 \leq i \leq n-2, 0 \leq j \leq n-2$,
$$\sum_{i'=1}^{i} a_{i',j+1} + \sum_{j'=1}^{j+1} a_{i+1,j'} - \sum_{i'=1}^{i} a_{i'j} \geq 0$$

Magog but not ASM

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

ASM but not Magog

$$\begin{pmatrix} {\color{red}0} & {\color{green}1} & 0 & 0 \\ {\color{red}1} & {\color{green}-1} & 1 & 0 \\ {\color{green}0} & {\color{green}0} & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

ASM and Magog

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Permutation magog matrices

Magog triangle

$$\begin{array}{ccccccccccc} & & & & & 5 & & & & & \\ & & & & 4 & & 5 & & & & \\ & & & 2 & & 4 & & 5 & & & \\ & & 2 & & 3 & & 4 & & 5 & & \\ & 2 & & 3 & & 4 & & 5 & & 6 & \\ 1 & & 2 & & 3 & & 4 & & 5 & & 6 \end{array} \longleftrightarrow \begin{array}{c} \text{Magog matrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Theorem (Holmlund and S. 2024)

The magog matrices of order n with no negative ones are the 132-avoiding permutation matrices.

Is there a pattern avoidance characterization of *all* magog matrices?

$\rightarrow$ New tool: Recent papers on key-avoidance and classical-avoidance in ASMs, joint with Bouvel and Smith as well as Egge and Troyka

ASMs / Magog matrices

Only ASM:
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Both:
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Only magog:
$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Refined enumerations: Boundary 1s

For both ASMs and Magog matrices:

The first row (or first column or last row) has one 1 and the rest 0s.

ASM properties

- The last column has one 1 and the rest 0s.
- # with a 1 in a specified corner is the number of ASMs of size $n-1$.
- Formula for # with a 1 in a specified entry of an outer row or column.

Magog matrix properties (Holmlund and S.)

- The last column can have -1s and more 1s.
- # with a 1 in the SW corner is the number of ASMs of size $n-1$.
- # with a 1 in the NW corner is 1.
- # with a 1 in the NE corner equals the number with a 1 in row 1 column $n-1$.
- # with a 1 in row 1 column 2 is $2^{n-1}-1$.
- # with a 1 in row 2 column 1 is $C_n - 1$.

Refined enumerations: Inversions

The *inversion number* of A is $\displaystyle\sum_{1\le k<i\le n}\sum_{1\le j<l\le n} A_{ij}A_{kl}$.

For both ASMs and Magog matrices:

- The identity matrix is the only one with no inversions.
- The anti-identity matrix is the only one with the maximum number of inversions: $\binom{n}{2}$.

ASM properties

- $\#$ with 1 inversion is $n-1$.
- $\#$ with 2 inversions is $\binom{n-1}{2} - 2(n-1) = \frac{(n-2)(n+3)}{2}$.
- $\#$ with $\binom{n}{2} - 1$ inversions is $\binom{n}{2}$

Magog matrix properties (HS 24)

- $\#$ with 1 inversion is 1.
- $\#$ with 2 inversions is $n+1$.
- Conj: $\#$ with $\binom{n}{2} - 1$ inversions is $2^n - n - 1$.

More conjectures involving *positive inversions*.

ASM vs. TSSCPP polytope

Definition (S. / Behrend-Knight)

The nth *alternating sign matrix polytope* ASM(n) is the convex hull in $\mathbb{R}^{n^2}$ of the $n \times n$ ASMs.

Definition (Holmlund and S.)

The nth *TSSCPP matrix polytope* TSSCPP(n) is the convex hull in $\mathbb{R}^{n^2}$ of all $n \times n$ magog matrices.

Theorem (S. 2009 / *BK 2007)

ASM(n) *satisfies:*

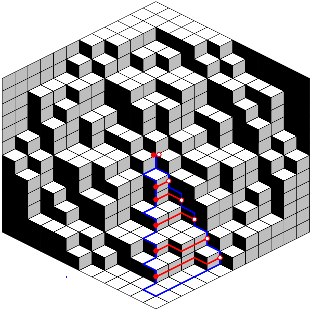
- ** Dimension* $(n-1)^2$
- ** Vertices are precisely the* $n \times n$ *ASMs*
- ** Inequality description*
- $4[(n-2)^2+1]$ *facets*
- *Nice face lattice description*
- *Projects to permutohedron*

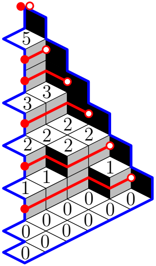
Theorem (Holmlund and S. 2024)

TSSCPP(n) *satisfies:*

- *Dimension at most* $(n-1)^2$
- *Vertices are precisely the set of* $n \times n$ *magog matrices*
- *Partial inequality description*

Known TSSCPP bijections

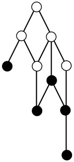




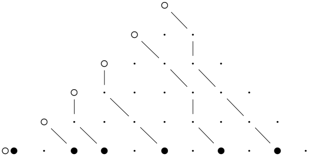
Magog triangle

6
4 5
3 4 5
2 3 5 6
1 2 3 4 6
1 2 3 4 5 6

Order ideal



Non-intersecting lattice paths



TSSCPP boolean triangles

Definition (S. 2018)

A *TSSCPP boolean triangle* of order n is a triangular integer array $\{b_{i,j}\}$ for $1 \leq i \leq n-1$, $n-i \leq j \leq n-1$ with entries in $\{0,1\}$ such that the diagonal partial sums satisfy $1 + \sum_{i=j+1}^{i'} b_{i,n-j-1} \geq \sum_{i=j}^{i'} b_{i,n-j}$.

Not a TSSCPP boolean triangle

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 0 & & 0 & & \\
 & 1 & & 0 & & 0 & & 1 & \\
0 & & 0 & & 0 & & 1 & & 0
\end{array}
$$

TSSCPP boolean triangles

Definition (S. 2018)

A *TSSCPP boolean triangle* of order n is a triangular integer array $\{b_{i,j}\}$ for $1 \leq i \leq n-1$, $n-i \leq j \leq n-1$ with entries in $\{0,1\}$ such that the diagonal partial sums satisfy $1 + \sum_{i=j+1}^{i'} b_{i,n-j-1} \geq \sum_{i=j}^{i'} b_{i,n-j}$.

A TSSCPP boolean triangle

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 0 & & 0 & & \\
 & 1 & & 0 & & 1 & & 1 & \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
$$

Boolean triangle polytope

Definition (Holmlund and S. 2024)

The nth *boolean triangle polytope* $\mathrm{BTP}(n)$ is the convex hull in $\mathbb{R}^{\binom{n}{2}}$ of all TSSCPP boolean triangles of order n.

Theorem (Holmlund and S. 2024)

$\mathrm{BTP}(n)$ *satisfies the following:*

- *Dimension* $\binom{n}{2}$.
- *Vertices are precisely the* TSSCPP *boolean triangles of order* n.
- *Inequality description*
- $\dfrac{(n-1)(3n-2)}{2}$ *facets*

The many faces of alternating sign matrices

ASM

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

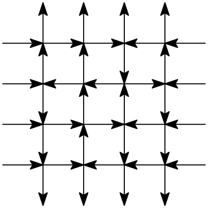
Monotone triangle

$$\begin{array}{ccccccc} & & & 3 & & & \\ & & 1 & & 4 & & \\ & 1 & & 3 & & 4 & \\ 1 & & 2 & & 3 & & 4 \end{array}$$

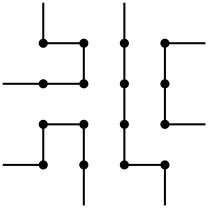
Height function

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 2 & 3 \\ 2 & 1 & 2 & 3 & 2 \\ 3 & 2 & 3 & 2 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

Six-vertex model



Fully-packed loop



Corner sum matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}$$

The many faces of magog matrices (Bansal and S. 2025+)

Magog matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

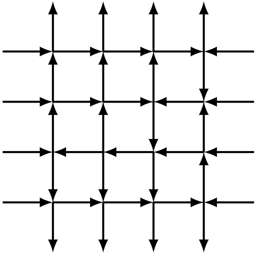
Magog triangle

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 3 & & 4 & & \\ & 1 & & 2 & & 3 & \\ 1 & & 2 & & 3 & & 4 \end{array}$$

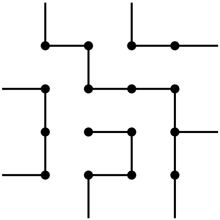
Magog height function

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 3 \\ 2 & 3 & 4 & 3 & 2 \\ 3 & 2 & 1 & 0 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

Magog vertex model

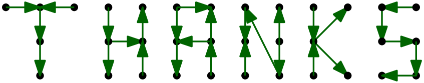


Magog fully-packed loop



Magog corner sum

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 & 3 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}$$



Main papers for this talk:

- R. Bansal and J. Striker, The many faces of magog matrices, *In Preparation*.
- V. Holmlund and J. Striker, Totally symmetric self-complementary plane partition matrices and related polytopes, *Annals of Combinatorics* (2024), 38 pp.

Related papers:

- M. Bouvel, E. Egge, R. Smith, J. Striker, and J. Troyka, Classical pattern avoidance in alternating sign matrices, `arXiv:2411.07662` (Submitted).
- M. Bouvel, R. Smith, and J. Striker, Key-avoidance for alternating sign matrices, *Discrete Math Theoretical Computer Science* **27** (2025), no. 1, 28 pp.
- D. Huang and J. Striker, A pipe dream perspective on totally symmetric self-complementary plane partitions, *Forum of Math, Sigma* **12** (2024), no. e17, 19 pp.
- J. Striker, Permutation totally symmetric self-complementary plane partitions, *Ann. Comb.* **22** (2018), no. 3, 641–671.
- J. Striker, The alternating sign matrix polytope. *Electronic Journal of Combinatorics,* **16** (2009), no. 1.